THE ALGEBRA OF FUNCTIONS

DEFINITIONS

1: $[f + g](x) = f(x) + g(x)$ for every x that is in both the domain of f and the domain of g.

2. $[f - g](x) = f(x) - g(x)$ for every x that is in the domain of f and the domain of g.

3. $[f \cdot g](x) = f(x) \cdot g(x)$ for every c that is in both the domain of f and in the domain of g.

4. $\left[\frac{f}{g}\right](x) = \frac{f(x)}{g(x)}$; $g(x) \neq 0$ for every x that is in both the domain of f and the domain of g.

5. $[f \circ g](x) = f(g(x))$ for every x that is in the domain of g such that g(x) is in the domain of f.

I. SUM, DIFFERENCE, PRODUCT AND QUOTIENT. When trying to perform these binary operations on <u>two</u> functions g(x) and f(x) we must make sure that the functions have the <u>same</u> domain. Once we have established this fact we simply add, subtract, multiply or divide the ranges of the functions. Study the following examples carefully.

Example 1.

$f(x) = \{(0,7),(3,4),(-1,2)\}$ $\quad g(x) = \{(0,5),(4,4),(-1,6)\}$

$[f + g](-1) = f(-1) + g(-1) = 2 + 6 = 8$

$[f - g](3) = f(3) - g(3)$ = This is not possible because g(3) does not exist.

$[f - g](0) = f(0) - g(0) = 7 - 5 = 2$

$[f \cdot g](-1) = f(-1) \cdot g(-1) = 2 \cdot 6 = 12$

Example 2 $\quad f(x) = x^2 + 3 \quad g(x) = x - 3$

$$[f + g](x) = f(x) + g(x) = (x^2 + 3) + (x - 3) = x^2 + x$$

$$[f \cdot g](x) = f(x) \cdot g(x) = (x^2 + 3)(x - 3) = x^3 - 3x^2 + 3x - 9$$

$$\left[\frac{f}{g}\right](x) = \frac{f(x)}{g(x)} = \frac{x^2 + 3}{x - 3}$$

Be careful --This is defined for all x except x = 3. W HY?

$$[f - g](x) = f(x) - g(x) = x^2 + 3 - (x - 3) = x^2 + 3 - x + 3 = x^2 - x + 6$$

PROBLEM SET I Find a) $[f + g](x)$ b) $[f - g](x)$ c) $[f \cdot g](x)$ and d) $\left[\frac{f}{g}\right](x)$ for the following functions. Then give the domain and range for all of the answers

1. $f = \{(1,2),(2,2),(3,5),(4,6)\}$ $g = \{(2,2),(-1,3),(3,0),(4,9)\}$
2. $g = \{(0,3),(2,6),(-1,5),(5,0)\}$ $h = \{(-1,7),(5,8),(2,3)(7,7)\}$
3. $h(x) = x^2$ $m(x) = x - 1$
4. $f(x) = x - 2$ $g(x) = x^2 + 7$
5. $p(x) = \{(0,3),(3,5),(1,6)(-3,7)\}$ $g(x) = x + 4$
6. $h(x) = \{(5,4),(7,3),(-1,6),(-3,0),(2,2)\}$ $f(x) = x^3 - 1$

SECTION II. COMPOSITES. In this section things get a bit harder. When finding the composite of any two functions we do not look for where the functions have the same domain but rather where the range of one function is in the domain of the other. This is a bit confusing but study the definition and the following examples.

Example 1. $f(x) = \{(5,4),(3,7),(4,4)\}$ $g(x) = \{(7,4),(4,2),(3,5)\}$

$[f \circ g](7) = f(g(7))$ since $g(7) = 4$ we substitute 4 into f.
$= f(4)$
$= 4$ The composite is $(7,4)$

$[f \circ g](4) = f(g(4))$
$= f(2)$ since $f(2)$ does not exist the composite cannot be found

$[f \circ g](3) = f(g(3))$ since $g(3) = 5$ we substitute 5 into f
$= f(5)$ now find $f(5)$ which is 4
$= 4$ The composite is $(3,4)$.

Example 2 $f(x) = \{(5,4),(3,7),(4,4)\}$ $g(x) = \{(7,4),(4,2),(3,5)\}$

$[g \circ f](5) = g(f(5))$ since $f(5) = 4$ we substitute 4 into g.
$= g(4)$
$= 2$ Composite is $(5,2)$

$[g \circ f](3) = g(f(3))$
$= g(7)$
$= 4$ the composite is $(3,4)$

$[g \circ f](4) = g(f(4))$
$= g(4)$
$= 2$ The composite is $(4,2)$

Example 3 $f(x) = x^2 + 2$ $g(x) = x - 2$

$[f \circ g](x) = f(g(x))$ since $g(x) = x - 2$ we substitute $x - 2$ into $f(x)$. Then find $f(x - 2)$.

$= f(x - 2)$

$= (x - 2)^2 + 2$

$= x^2 - 4x + 6$

$[g \circ f](x) = g(f(x))$

$= (\qquad) - 2$ substitute $(x^2 + 2)$ into $g(x)$

$= (x^2 + 2) - 2$

$= x^2$

PROBLEM SET II

1. $f = \{(6,0),(4,3),(5,9),(1,4)\}$ $g = \{(0,3),(2,1),(9,6)\}$

   $[f \circ g]$ = ____________________

   $[g \circ f]$ = ____________________

2. $g = \{(3,5),(-1,6),(4,3),(2,7),(-2,9)\}$

   $h = \{(7,-1),(9,5),(-3,-5),(5,2),(3,7)\}$

   $[g \circ h]$ = ____________________

   $[h \circ g]$ = ____________________

3. $p(x) = x^2 + 4$ $h(x) = x - 5$

   $[p \circ h](x)$ = ____________________

   $[h \circ p](x)$ = ____________________

4. $g(x) = x^3 - 2$ $h(x) = x^2 + 1$

   $[g \circ h](x)$ = ____________________

   $[h \circ g](x)$ = ____________________

5. $p(x) = 3x + 2$ $g(x) = x^2 - 4$

   $[p \circ g](x)$ = ____________________

   $[g \circ p](x)$ = ____________________

6. $f(x) = x - 2$ $g(x) = x^2 + 7$

   $[f \circ g](x)$ = ____________________

   $[g \circ f](x)$ = ____________________

7. $f = \{(2,1),(3,4),(4,4),(5,7),(-8,4),(0,1)\}$ $g(x) = 2x + 3$

$[f \circ g] =$ ______________________

$[g \circ f] =$ ______________________

8. Using your answers from the above problems fill in the following chart.

		Domain	Range
1.	f o g	______	______
	g o f	______	______
2.	g o h	______	______
	h o g	______	______
3.	p o h	______	______
	h o p	______	______
4.	g o h	______	______
	h o g	______	______
5.	p o g	______	______
	g o p	______	______
6.	f o g	______	______
	g o f	______	______
7.	f o g	______	______
	g o f	______	______

III. In this section we are concerned with finding the sum, difference, product, quotient and composite of any two functions at a given point. To do this merely evaluate the two functions at the particular point and then perform the indicated operation. We shall also include inverses in this section so check your functions 1 a p for a quick review.

Example 1. $f(x) = 2x^2 - 3$ $\quad g(x) = 3x + 7$

$$[f + g](4) = f(4) + g(4) = 29 + 19 = 48$$

$f(4) = 2(4)^2 - 3 = 29$

$g(4) = 3(4) + 7 = 19$

$$\left[\frac{f}{g}\right](-1) = \frac{f(-1)}{g(-1)} = \frac{2(-1)^2 - 3}{3(-1) + 7} = \frac{-1}{4}$$

$$[f \circ g](-1) = f(g(-1)) = f(4) = 29$$

$$[g \circ f](3) = g(f(3)) = g(15) = 52$$

PROBLEM SET III

1. If $f(x) = 2x^2 + 6$ and $g(x) = 7x + 2$; then find

a) $[g \circ f](4)$ = ________ f) $[g \circ g^{-1}](x)$ ________

b) $[g \circ g](2)$ = ________ g) $[f \circ g](0)$ = ________

c) $[g \circ f](5)$ = ________ h) $[f \circ g](x)$ = ________

d) $[f \circ f](0)$ = ________ i) $[f^{-1} \circ g](x)$ = ________

e) $[g^{-1} \circ g](200)$ = ________ j) $\left[\frac{f}{g}\right](\frac{-2}{7})$ = ________

k) $\left[\frac{g}{f}\right](0)$ = ________

2. $f = \{(2,1),(3,4),(4,4),(5,7),(-8,4),(0,1)\}$

$g(x) = x - 1$

a) $[f + g](2)$ = ________ d) $\left[\frac{f}{g}\right](0)$ = ________

b) $[f - g](5)$ = ________ e) $[f \circ g](5)$ = ________

c) $[f \cdot g](4)$ = ________ f) $[g \circ f](2)$ = ________

3. If $f = \{(1,1),(2,4),(3,9),(-1,7),(-2,5),(-3,6)\}$ and

$g = \{(1,4),(2,1),(4,5),(-2,0)\}$ then

f^{-1} = ________

g^{-1} = ________

a) the domain of $f + g$ is ________

b) the domain of $\frac{f}{g}$ is ________

c) $[f^{-1} \circ f]$ = ________

d) $[g^{-1} \circ g]$ = ________

e) What statement can you make about any function composite its inverse? ________

4. Let $f(x) = 3x - 7$ and $g(x) = 2x + k$

Find k so that $[f \circ g](x) = [g \circ f](x)$

5. Complete the following table

Given	Domain	Range
$f(x) = x^3 + 8$		
$g(x) = x - 7$		
$h(x) = [\![x]\!]$		
$j(x) = \lvert x \rvert$		
$k(x) = \frac{1}{x}$		

		Domain	Range
$l(x) = \log_2 x$			
$m(x) = x^2 + 2x + 7$			
$[h \circ g](x) =$			
$[j \circ h](x) =$			
$[j \circ k](x) =$			
$[k \circ l](x) =$			
$[j \circ l](x) =$			
$[l \circ j](x) =$			
$[j \circ m](x) =$			

NOW: 1. Check over the behavioral objectives ---page 7.

2. Take the trial run

3. Pass the test

ALGEBRA OF FUNCTIONS

BEHAVIORAL OBJECTIVES

I. Define each of the following binary operations

A. $[f + g](x)$

B. $[f - g](x)$

C. $[f \cdot g](x)$

D. $\left[\frac{f}{g}\right](x)$

E. $[f \circ g](x)$

II. Given two functions f and g, determine

A. f + g; its domain and range

B. f − g; its domain and range

C. f · g; its domain and range

D. $\frac{f}{g}$; its domain and range

E. f ∘ g; its domain and range

ALGEBRA OF FUNCTIONS ANSWERS

PROBLEM SET 1

1.	$f + g = \{(2,4), (3,5), (4,15)\}$	$D = \{2,3,4\}$	$R = \{4,5,15\}$
	$f - g = \{(2,0), (3,5), (4,-3)\}$	$D = \{2,3,4\}$	$R = \{0,5,-3\}$
	$f \cdot g = \{(2,4), (3,0), (4,54)\}$	$D = \{2,3,4\}$	$R = \{4,0,54\}$
	$f / g = \{(2,1), (4,\frac{2}{3})\}$	$D = \{2,4\}$	$R = \{1,\frac{2}{3}\}$
2.	$g + h = \{(2,9), (-1,12), (5,8)\}$	$D = \{2, -1, 5\}$	$R = \{8, 9, 12\}$
	$g - h = \{(2,3), (-1,-2), (5,-8)\}$	$D = \{-1, 2, 5\}$	$R = \{-8, -2, 3\}$
	$g \cdot h = \{(2,18), (-1,35), (5,0)\}$	$D = \{-1, 2, 5\}$	$R = \{0, 18, 35\}$
	$g / h = \{(2,2), (-1,\frac{5}{7}), (5,0)\}$	$D = \{-1, 2, 5\}$	$R = \{0, \frac{5}{7}, 2\}$
3.	$[h + m](x) = x^2 + x - 1$	D = all reals	$R = \{y: y \geq -\frac{5}{4}\}$
	$[h - m](x) = x^2 - x + 1$	D = all reals	$R = \{y: y \geq \frac{3}{4}\}$
	$[h \cdot m](x) = x^3 - x^2$	D = all reals;	R = all reals
	$[h / m](x) = \frac{x^2}{x - 1}$	$D = x: x \neq 1$	*$R = \{y: y \leq 0$ or $y \geq 4\}$
4.	$[f + g](x) = x^2 + x + 5$	D = all reals	$R = \{y: y \geq \frac{19}{4}\}$
	$[f - g](x) = -x^2 + x - 9$	D = all reals	$R = \{y: y \leq \frac{-35}{4}\}$
	$[f \cdot g](x) = x^3 - 2x^2 + 7x - 14$	D = all reals	R = all reals
	$[f / g](x) = \frac{x - 2}{x^2 + 7}$	D = all reals	*$R = \{y: -.4 \leq y \leq .9\}$
5.	$p + g = \{(0,7), (3,12), (1,11), (-3,8)\}$	$D = \{-3,1,0,3\}$	$R = \{7,12,11,8\}$
	$p - g = \{(0,-1), (3,-2), (1,1), (-3,6)\}$	$D = \{-3,1,0,3\}$	$R = \{-1,-2,1,6\}$
	$p \cdot g = \{(0,12), (3,35), (1,30), (-3,7)\}$	$D = \{-3,1,0,3\}$	$R = \{12,35,30,7\}$
	$p / g = \{(0,\frac{3}{4}), (3,\frac{5}{7}), (1,\frac{6}{5}), (-3,7)\}$	$D = \{-3,1,0,3\}$	$R = \{\frac{3}{4}, \frac{5}{7}, \frac{6}{5}, 7\}$

* Only by checking a great number of values could you determine the range of this function. The range for f/g in #4 is approximate. Methods for determing the range of functions like these will be discussed in the last quin of Math Analysis and even more thoroughly in Calculus.

6. $h + f = \{(5,128),(7,345),(-1,4),(-3,-28),(2,9)\}$ $D = \{5,7,-1,-3,2\}$
$R = \{128,345,4,-28,9\}$

$h - f = \{(5,-120),(7,-339),(-1,8),(-3,28),(2,-5)\}$ $D = \{-3,-1,2,5,7\}$
$R = \{-120,-339,8,28,-5\}$

$h \cdot f = \{(5,496),(7,1026),(-1,-12),(-3,0),(2,14)\}$ $D = \{-3,-1,2,5,7\}$
$R = \{496,1026,-12,0,14\}$

$h / f = \{(5,\frac{1}{31}),(7,\frac{1}{114}),(-1,-3),(-3,0),(2,\frac{2}{7})\}$ $D = \{-3,-1,2,5,7\}$
$R = \{\frac{1}{31}, \frac{1}{114}, -3, 0, \frac{2}{7}\}$

PROBLEM SET II

1. $\{(2,4), (9,0)\}$
$\{(6,3), (5,6)\}$

2. $\{(7,6), (5,7)\}$
$\{(3,2),(4,7),(-2,5),(2,-1)\}$

3. $x^2 - 10x + 29$
$x^2 - 1$

4. $(x^2 + 1)^3 - 2$
$(x^3 - 2)^2 + 1$

5. $3(x^2 - 4) + 2$
$(3x + 2)^2 - 4$

6. $x^2 + 5$
$(x - 2)^2 + 7$

7. $\{(-\frac{1}{2},1),(0,4),(\frac{1}{2},4),(1,7),(-\frac{11}{2},4),(-\frac{3}{2},1)\}$
$\{(2,5),(3,11),(4,11),(5,17),(-8,11),(0,5)\}$

8.

	DOMAIN	RANGE		DOMAIN	RANGE
1.	$\{2, 9\}$	$\{4, 0\}$	2.	$\{7, 5\}$	$\{6, 7\}$
	$\{6, 5\}$	$\{3, 6\}$		$\{3,4,-2,2\}$	$\{2,7,5,-1\}$
3.	Reals	$\{y: y \geq 4\}$	4.	Reals	$\{y: y \geq -1\}$
	Reals	$\{y: y \geq -1\}$		Reals	$\{y: y \geq 1\}$
5.	Reals	$\{y: y \geq -10\}$	6.	Reals	$\{y: y \geq 5\}$
	Reals	$\{y: y \geq -4\}$		Reals	$\{y: y \geq 7\}$

7. $\{-\frac{1}{2},0,\frac{1}{2},1,-\frac{11}{2},-\frac{3}{2}\}$ $\{1,4,7\}$
$\{2,3,4,5,-8,0\}$ $\{5,11,17\}$

PROBLEM SET III

1. a) 268 b) 114 c) 394 d) 78 e) 200 f) x g) 14
h) $2(7x + 2)^2 + 6$ i) Undefined, f^{-1} does not exist j) Undefined
k) $\frac{1}{3}$

2. a) 2; b) 3; c) 12; d) -1; e) 4 f) 0

3. $f^{-1} = \{(1,1),\ (4,2),\ (9,3),\ (7,-1),\ (5,-2),\ (6,-3)\}$

$g^{-1} = \{(4,1),\ (1,2),\ (5,4),\ (0,-2)\}$

a) $\{1,\ 2,\ -2\}$ b) $\{1,\ 2\}$ c) $\{(1,1),(2,2),(3,3),(-1,-1),(-2,-2),(-3,-3)\}$

d) $\{(1,1),(2,2),(4,4),(-2,-2)\}$

4. $k = -\frac{7}{2}$

5.

	DOMAIN	RANGE
	Reals	Reals
	Reals	Reals
	Reals	Integers
	Reals	$\{y:\ y \geq 0\}$
	$\{x:\ x \neq 0\}$	$\{y:\ y \neq 0\}$
	$\{x:\ x > 0\}$	Reals
	Reals	$\{y:\ y \geq 6\}$
$[\![x - 7]\!]$	Reals	Integers
$\lvert [\![x]\!] \rvert$	Reals	Non-negative integers
$\left\lvert \frac{1}{x} \right\rvert$	$\{x:\ x \neq 0\}$	$\{y:\ y > 0\}$
$\frac{1}{\log_2 x}$	$\{x:\ x > 0,\ x \neq 1\}$	$\{y:\ y \neq 0\}$
$\lvert \log_2 x \rvert$	$\{x:\ x > 0\}$	$\{y:\ y \geq 0\}$
$\log_2 \lvert x \rvert$	$\{x:\ x \neq 0\}$	Reals
$\lvert x^2 + 2x + 7 \rvert$	Reals	$\{y:\ y \geq 6\}$

ALGEBRA OF FUNCTIONS

TRIAL RUN

I. Define:

(a) $[f + g](x)$

(b) $[f \circ g](x)$

(c) $\left[\frac{f}{g}\right](x)$

II. Suppose $f(x) = 4x + 4$, $g(x) = [\![x]\!]$, $h(x) = |x + 2|$, $j(x) = x^2$,
$k(x) = \{(3,3), (2,0), (4,1)\}$, $m(x) = \{(1,8), (0,4), (3,0)\}$

(a) Evaluate:

1. $[f + k](4) =$
2. $[f - h](-1) =$
3. $[j + f](2) =$
4. $[k + m](3) =$
5. $\left[\frac{k}{m}\right](3) =$
6. $\left[\frac{f}{h}\right](1) =$
7. $\left[\frac{f}{h}\right](0) =$
8. $[f \circ k](4) =$
9. $[m \circ m](0) =$
10. $j(f(0)) =$
11. $[f \circ h](2) =$
12. $[f \circ g](6.2) =$

(b) Find:

1. $[j + f](x) =$
2. $[k \circ m] =$
3. $[g \circ f](x) =$
4. $[g \circ h](x) =$
5. $[m \circ h]$
6. $\left[\frac{h}{g}\right](x) =$

(c) Determine the domain and range of each of numbers 1 through 5 of (b) above.

I. (a) $[f + g](x) = f(x) + g(x)$ for every x in the domains of f and g.

(b) $[f \circ g](x) = f(g(x))$ for every x in the domain of g such that g(x) is in the domain of f.

(c) $\left[\frac{f}{g}\right](x) = \frac{f(x)}{g(x)}$ for every x in the domains of f and g such that $g(x) \neq 0$.

II. (a)

1. 21	2. -1	3. 16
4. 3	5. undefined	6. 8/3
7. 2	8. 8	9. undefined
10. 16	11. 20	12. 28

(b) 1. $x^2 + 4x + 4$ 2. $\{(0, 1)\}$ 3. $[\![4x + 4]\!]$

4. $[\![|x + 2|]\!]$ 5. $\{(-3,8),(-1,8),(-2,4),(1,0),(-5,0)\}$

6. $\frac{|x + 2|}{[\![x]\!]}$

(c) 1. Domain = $\{x: x \in R\}$ Range = $\{y: y \geq 0\}$

2. Domain = $\{0\}$ Range = $\{1\}$

3. Domain = $\{x: x \in R\}$ Range = $\{y: y \in \text{Integers}\}$

4. Domain = $\{x: x \in R\}$ Range = $\{y: y \text{ is a non-negative integer}\}$

5. Domain = $\{-3, -1, -2, 1, -5\}$ Range = $\{8, 4, 0\}$