

BEHAVIORAL OBJECTIVES

I. Identify

- A. $>$ greater than
- B. \geq greater than or equal to
- C. $<$ less than
- D. \leq less than or equal to

II. Use the primary inequality theorems

- A. If $a > b$ then $a + c > b + c$
- B. If $a > b$ and $c > 0$, then $ac > bc$
- C. If $a > b$ and $c < 0$, then $ac < bc$
- D. If $|a| > b$ then $a > b$ or $a < -b$ (For $b > 0$)
- E. If $|a| < b$ then $-b < a < b$ (for $b > 0$)

III. Solve and graph the solution of an inequality involving

- A. One first degree variable
- B. A quadratic expression in one variable
- C. A product and/or quotient of several binomials in one variable
- D. An absolute value expression in one variable

IV. Graph inequalities involving two variables where the equality graph is a(n)

- A. Line
- B. Two intersecting lines
- C. Circle
- D. Ellipse
- E. Parabola
- F. Hyperbola

INEQUALITIES

The set of real numbers is ordered; i.e. given two real numbers a and b, such that $a \neq b$, then $a > b$ (a is greater than b), or $b > a$ (b is greater than a.) If $a > b$ and $b > c$, then $a > c$. This property is called the transitive property for inequalities.

This L.A.F. will concern itself with the solution of various types of inequalities. Most techniques employed to solve equations may be used to solve inequalities. One major exception is that of multiplying or dividing both sides of an inequality by a negative.

Example: $7 > 2$ (7 is greater than 2)
 $-7 < -2$ (-7 is less than -2)

NOTICE: Multiplication of both sides of an inequality by a negative reverses the inequality sign.

Vocabulary: $<$ less than
 \leq less than or equal to
 $>$ greater than
 \geq greater than or equal to

Initial Basic Theorems:

1. If $a > b$ then $a + c > b + c$
2. If $a > b$ and $c > 0$, then $ac > bc$
3. If $a > b$ and $c < 0$, then $ac < bc$
4. If $a > b$, and $b > c$, then $a > c$.

SECTION I

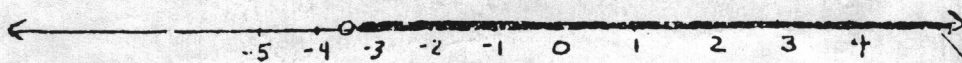
FIRST DEGREE INEQUALITIES IN ONE VARIABLE

Consider the inequality: $4x + 8 > 2x + 1$. To solve the inequality proceed as you would with the equation: $4x + 8 = 2x + 1$.

Solution: $4x + 8 > 2x + 1$
 $4x > 2x - 7$ (Subtract 8 from both sides. Theorem 1)
 $2x > -7$ (Subtract $2x$ from both sides. Theorem 1)
 $x > -\frac{7}{2}$ (Multiply both sides by $\frac{1}{2}$. Theorem 3.)

The solution set is: $\{x: x > -\frac{7}{2}\}$. The solution set can also be written $(-\frac{7}{2}, \infty)$

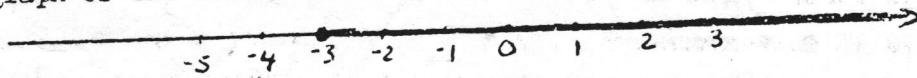
The graph of the solution is:



Example 2 Solve the inequality: $5x - 7 \leq 8x + 2$

Solution: $5x - 7 \leq 8x + 2$
 $5x \leq 8x + 9$
 $-3x \leq 9$
 $x \geq -3$

The solution set is: $\{x: x \geq -3\}$. This can be written: $[-3, \infty)$
 The graph of the solution set is:

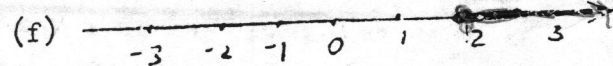
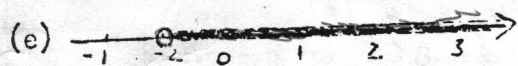
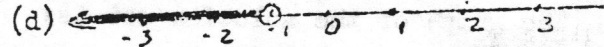
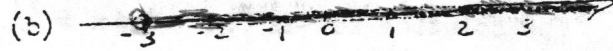


The strange way to write the solution set is given as an alternate form, just might need some further clarification. Study the following examples:

1. $(3, \infty)$ means $x > 3$
2. $[3, \infty)$ means $x \geq 3$
3. $(-\infty, 3)$ means $x < 3$
4. $(-\infty, 3]$ means $x \leq 3$

EXERCISE 1

1. Write an inequality which identifies each graph sketched below:



2. Sketch a graph of each of the following:

- | | | |
|--------------------|--------------------|--------------------|
| (a) $(-4, \infty)$ | (b) $x > 4$ | (c) $x \leq 5$ |
| (d) $[5, \infty)$ | (e) $[-2, \infty)$ | (f) $(-\infty, 4]$ |

3. Solve and graph each of the following:

- | | | |
|--|--|---|
| (a) $7x - 1 \leq 2x + 4$ | (b) $-4x + 3 \geq x - 7$ | (c) $\frac{x+7}{4} \geq 0$ |
| (d) $\frac{7x-1}{2} < \frac{x+1}{4}$ | (e) $3(x+8) < 2x-1$ | (f) $\frac{1}{3}x + 2 > \frac{1}{4}x - 8$ |
| (g) $\frac{10+2x}{3} \leq \frac{7}{5}$ | (h) $\frac{8x-3}{7} \geq \frac{2x}{3}$ | |

COMPOUND INEQUALITIES

SECTION II

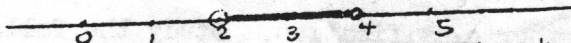
This section is much like the previous one with the exception that we have two inequality signs to worry about. Study the following examples carefully:

Example 1: Find the solution set of the inequality: $5 < 3x - 1 \leq 11$

Solution: $5 < 3x - 1 \leq 11$
 $6 < 3x \leq 12$
 $2 < x \leq 4$

$-4 < 5x \leq 15$

The solution set is $\{x: 2 < x \leq 4\}$. This can be written: $(2, 4]$

The graph of the solution is: 
The solution set is all real numbers between 2 and 4, including 4.

Example 2:

Find the solution set of the inequality: $-1 \leq -\frac{1}{2}x + 2 \leq 4$

Solution: $-1 \leq -\frac{1}{2}x + 2 \leq 4$
 $-3 \leq -\frac{1}{2}x \leq 2$
 $6 \geq x \geq -4$

The solution set is $\{x: -4 \leq x \leq 6\}$. This can be written: $[-4, 6]$

The graph of the solution set is:



Example 3:

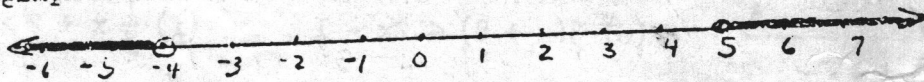
Find the solution set of the inequality: $x + 8 < 4$ or $x - 3 \geq 2$

Solution: $x + 8 < 4$ or $x - 3 \geq 2$
 $x < -4$ or $x \geq 5$

The solution set is: $\{x: x < -4$ or $x \geq 5\}$.

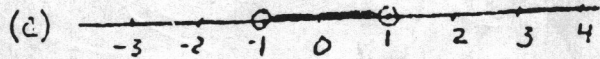
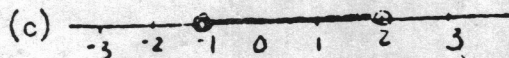
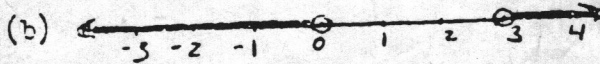
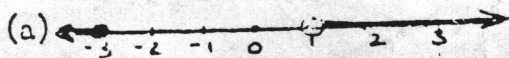
This solution set can be written: $(-\infty, -4) \cup [5, \infty)$

The graph of the solution set is two rays:



EXERCISE 2

1. Write an inequality which identifies each graph sketched below:



2. Sketch a graph of each of the following:

(a) $2 < x \leq 4$

(b) $[7, 8)$

(c) $x > 0$ or $x \leq -2$

(d) $(-\infty, 3] \cup (5, 7)$

(e) $-4 \leq x \leq 4$

(f) $[-2, 2]$

3. Solve each of the following inequalities:

(a) $x + 8 < 4$ or $x - 3 > 2$

(b) $10 < x + 1 < 12$

(c) $2x - 8 > 6$ or $3x + 1 < 10$

(d) $-4 < 2 - 9x < 5$

(e) $-7 < 5x - 3 \leq 12$

(f) $0 \leq 4x - 6 \leq 2$

(g) $-1 < \frac{3 - 7x}{4} \leq 6$

(h) $-\frac{1}{2}x - 1 > 2$ or $2x + 8 > 10$

(i) $5x - 1 > 14$ or $x + 2 < 1$

(j) $x + 1 \leq 3x + 2 < 5 + x$

(k) $2x + 3 \leq 5x - 7 \leq 2x + 8$

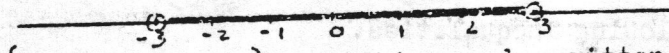
(l) $5 - x < -x + 3$

SECTION III

ABSOLUTE VALUE INEQUALITIES IN ONE VARIABLE

This section is mighty important. Its methods will be used in subsequent sections of this L.A.P. as well as in your later work in mathematics.

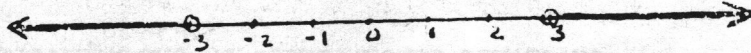
Consider the inequality $|x| < 3$. Both 3 and -3 are critical numbers for this expression. While neither 3 nor -3 actually solve the inequality, these numbers serve as boundary points. As long as the value of x is between 3 and -3, the inequality holds.

Graphically the solution set is:  Algebraically the solution set is $\{x: -3 < x < 3\}$, or this can be written: $(-3, 3)$.

A less than absolute value inequality generates a between statement solution.

Consider the inequality $|x| > 3$. Here again 3 and -3 are critical values. A solution value for x must be a number greater than 3 or less than -3.

For example: $|-10| > 3$; $|20| > 3$.

Graphically the solution set is:  Algebraically the solution set is $\{x: x < -3$ or $x > 3\}$. Or: $(-\infty, -3) \cup (3, \infty)$

A greater than absolute value inequality generates an or statement solution.

Study the following sample problems:

Example 1: Solve the inequality $|x - 8| > 4$

Solution: $x - 8 < -4$ or $x - 8 > 4$
 $x < 4$ or $x > 12$

Example 2: Solve the inequality: $|2x - 3| < 6$

Solution: $-6 < 2x - 3 < 6$
 $-3 < 2x < 9$
 $-\frac{3}{2} < x < \frac{9}{2}$

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Example 3: Solve the inequality: $|6x + 1| < 0$

Solution: No value of x satisfies this inequality. An absolute value expression is always ≥ 0 .

Example 4: Solve the inequality: $5|6x - 8| \geq 10$

Solution: $|6x - 8| \geq 2$
 $6x - 8 \leq -2$ or $6x - 8 \geq 2$
 $6x \leq 6$ or $6x \geq 10$
 $x \leq 1$ or $x \geq \frac{5}{3}$

EXERCISE 3

Solve each of the following inequalities:

1. $|x - 7| \geq 4$

2. $|2x + 1| < 5$

3. $|\frac{1}{2}s - 4| \leq 2$

4. $2|x - 3| \geq 8$

5. $|7x - 1| \leq -8$

6. $-7|x - 5| < 0$

7. $|\frac{x+2}{3}| < 5$

8. $|3x - 1| \geq 0$

9. $|3 - 11x| > 41$

10. $|25x - 8| \geq 7$

11. $|x| < -3$

12. $-|x| < -3$

SECTION IV

QUADRATIC INEQUALITIES IN ONE VARIABLE

First of all, let us consider a simple quadratic inequality such as $(x + 2)^2 \geq 9$.

The inexperienced would probably simply take the square root of both sides and

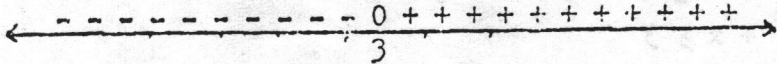
have the inequality: $x + 2 \geq 3$. However, this does not supply the complete solution.

Recall the much neglected theorem from algebra: For all x , $\sqrt{x^2} = |x|$.

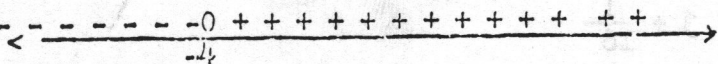
Hence the inequality under consideration would have the solution $|x + 2| \geq 3$.

This inequality is in turn solved giving us: $x \geq 1$ or $x \leq -5$.

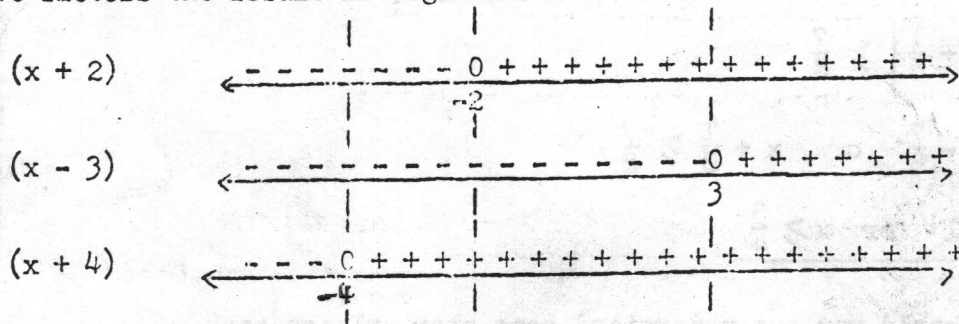
For the second factor: $(x - 3)$, $x = 3$ is the zero value. Its positive/negative graph is:



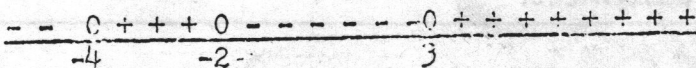
For the third factor, $(x + 4)$, $x = -4$ is the zero value. Its positive/negative graph is:



Now consider the graphs together. Keep in mind that if there is an even number of negative factors the result is positive, and if there is an odd number of negative factors the result is negative.

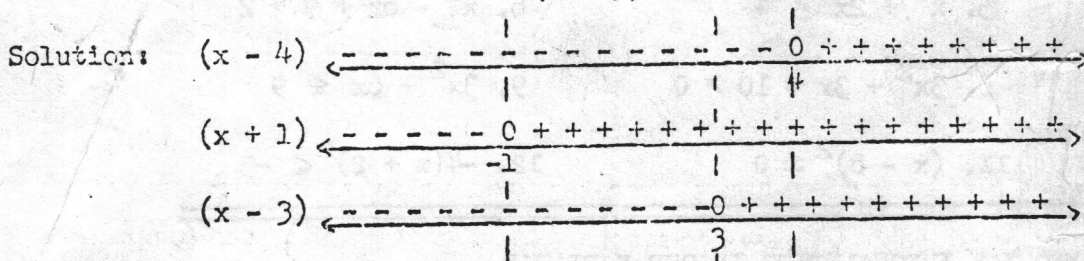


The product result graph is:



Thus the solution for $(x + 2)(x - 3)(x + 4) > 0$ is $x > 3$ or $-4 < x < -2$

Example 2: Solve the inequality: $\frac{(x - 4)(x + 1)}{(x - 3)} \leq 0$



A negative/zero result occurs for $3 \leq x \leq 4$ or $x \leq -1$. Since we cannot allow a denominator to be zero we must reject $x = 3$.

Hence the solution is: $3 < x \leq 4$ or $x \leq -1$

Example 3: Solve the inequality: $\frac{4}{x} \geq 10$

On first impulse, one would be inclined to multiply both sides of the inequality by x . However, x is a hidden number and we can't be sure whether it is positive or negative. Should it be negative of course, we would have to reverse the inequality sign. A way out is to multiply both sides by x^2 . x^2 is always greater than or equal to zero. In any case, we have to reject the zero as a solution because of x

occurring in the denominator.

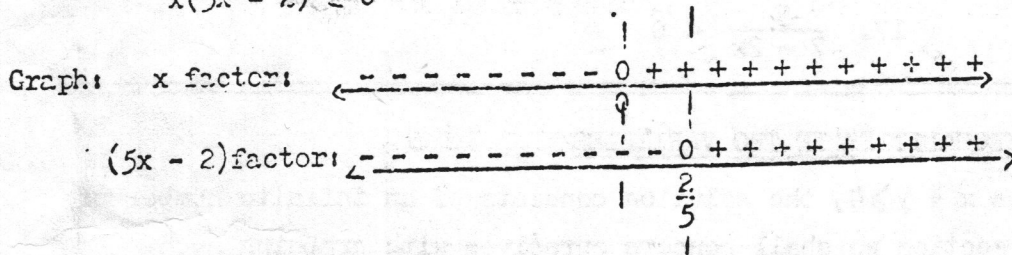
Solution: $x^2(\frac{4}{x}) \geq 10x^2$

$4x \geq 10x^2$

$10x^2 - 4x \leq 0$

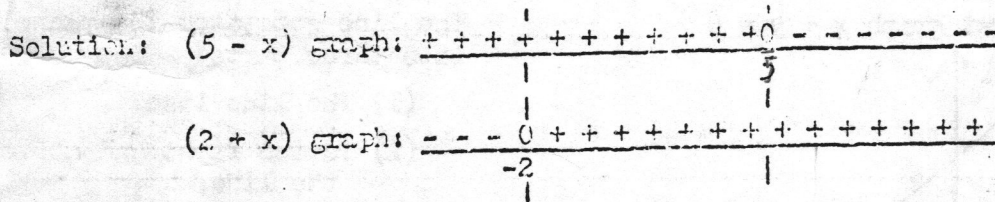
$2x(5x - 2) \leq 0$

$x(5x - 2) \leq 0$



For a negative product and $x \neq 0$ we choose $0 < x \leq \frac{2}{5}$ as the solution.

Example 4: Solve the inequality: $(5 - x)(2 + x) \geq 0$



For a product greater than or equal to zero we choose $-2 \leq x \leq 5$ as the solution.

Example 5: Solve the inequality: $(x - 4)^2(x + 2)(x - 8) \geq 0$

Solution: Since $(x - 4)^2$ is always greater than or equal to zero, we need only concern ourselves with the remaining two graphs. Sketch the graphs as in previous examples. Show that the solution is: $x \leq -2$ or $x \geq 8$.

EXERCISE 5

Solve each of the following inequalities:

1. $x^2 \geq 13x$

2. $(x - 4)(x + 2)^2 \geq 0$

3. $\frac{(x + 1)(x - 2)}{x + 4} \leq 0$

4. $(x - 7)(x + 3)(x - 3) \leq 0$

5. $(7 - x)(2 + x) > 0$

6. $x^3 + 7x^2 + 6x < 0$

7. $x^4 + 4x^2 + 4 > 0$

8. $\frac{2}{x^3} < \frac{1}{x^2}$

9. $\frac{3x+2}{2x-7} \leq 0$

10. $\frac{-x}{x^2 - x - 12} \geq 0$

11. $(x-9)^2(2x+13)^3 < 0$

12. $2x^2 - 9x + 7 < 0$

13. $\frac{1}{x^2} < 100$

14. $x^2 - 10x \leq 200$

15. $x^2 > 10x - 25$

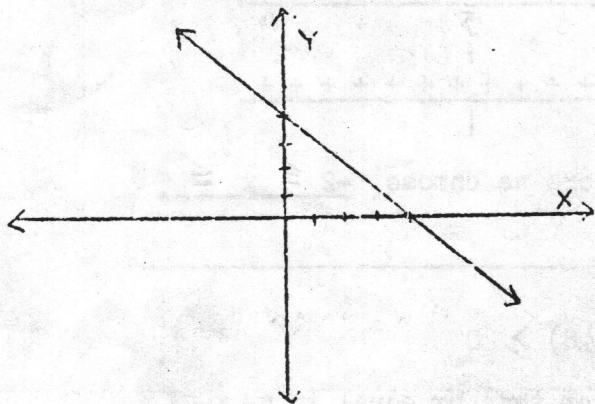
16. $\frac{2x^2 - 3x - 20}{x+3} < 0$

17. $\frac{5}{7-2x} < 0$

SECTION VI.INEQUALITIES IN TWO VARIABLES

For an inequality such as $x + y \geq 4$, the solution consists of an infinite number of ordered pairs. In this section we shall concern ourselves with graphing such inequalities on the X-Y plane. An inequality graph usually requires shading of part of the plane.

Example 1. For the inequality $x + y \geq 4$, we first graph $x + y = 4$



The line separates the plane into three parts:

- (1) The line itself.
- (2) To the right and above the line.
- (3) To the left and below the line.

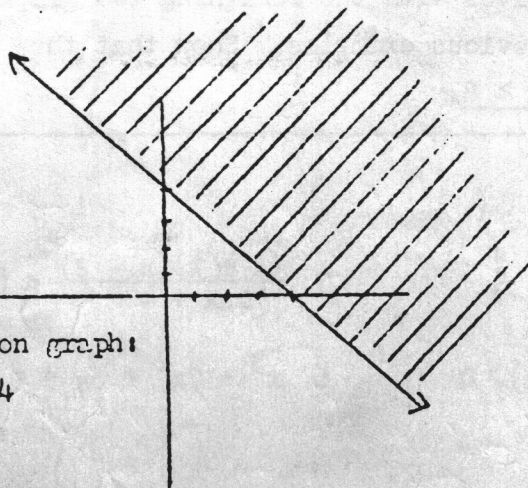
The points on the line satisfy the $x + y = 4$ part of the inequality.

What about the $x + y > 4$ part?

It will be one side of the line or the other. Test a point on one side of the line such as (0,0).

Notice, $0 + 0 \not\geq 4$. So we conclude that the solution set is the set of points on the other side of the line.

Should you question this, you could very well check some points on the shaded side and notice that they do indeed satisfy the inequality.

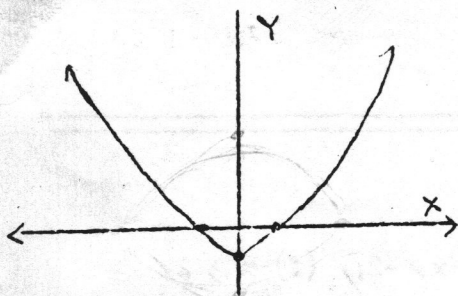


The solution graph:

$$x + y \geq 4$$

Example 2: Graph the inequality: $x^2 < y + 1$.

The critical equation is $x^2 = y + 1$.
This graphs as a parabola.

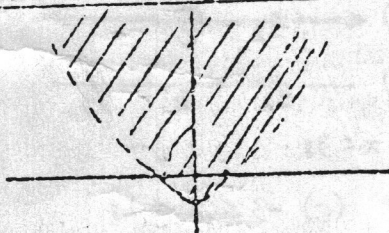


The parabola separates the plane into three parts:

- (1) The parabola
- (2) Inside the parabola
- (3) Outside the parabola.

Again, test a point which is not on the parabola. Take, for instance, (0,0). Notice, (0,0) does satisfy the inequality. Since (0,0) is inside the parabola, we shade the inside as the solution region.

The solution graph is:



Note: The parabola itself is dashed to indicate that the parabola is not part of the solution set.

Before you begin the exercises of this section you might want to become reacquainted with some general equations and their graphs.

- | | | |
|-------------------------------------|---------------------------------|-------------------------|
| 1. $x^2 + y^2 = 4$ (circle) | 2. $x^2 = y$ (parabola) | 3. $y^2 = x$ (parabola) |
| 4. $4x^2 + y^2 = 1$ (ellipse) | 5. $4x^2 - y^2 = 1$ (hyperbola) | |
| 6. $x^2 = y^2$ (intersecting lines) | 7. $ x = y$ V shaped curve | |

EXERCISE 6:

Graph each of the following inequalities:

- | | | |
|-----------------------|----------------------|-------------------------|
| 1. $x^2 + y^2 \leq 9$ | 2. $x^2 \geq y$ | 3. $9x^2 + y^2 \leq 81$ |
| 4. $ x \leq 2 y $ | 5. $3x + y > 0$ | 6. $4x^2 - y^2 > 4$ |
| 7. $ x > 4y$ | 8. $x^2 + 4y^2 > 16$ | 9. $5x > y$ |
| 10. $2y^2 < x - 4$ | 11. $ y \leq 2x$ | 12. $x^2 + y^2 \geq 4$ |

SECTION VII

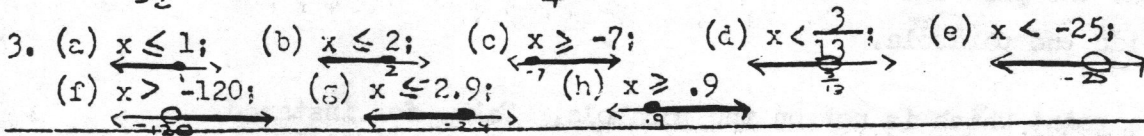
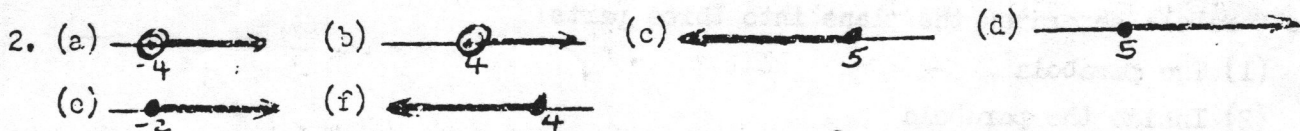
EVALUATION

1. Review the Behavioral Objectives.
2. Take the Trial Run.
3. Take the Test on this L.A.P.

ANSWERS

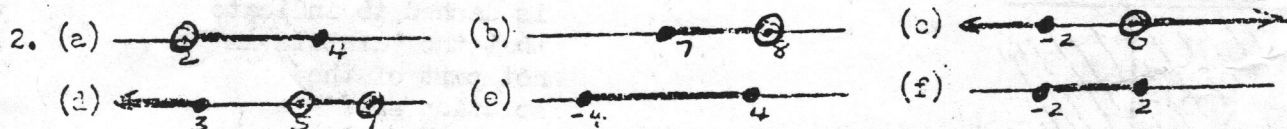
Exercise 1

1. (a) $x > 0$; (b) $x > -3$; (c) $x < 1$; (d) $x < -1$; (e) $x > -2$; (f) $x \geq 2$.



Exercise 2

1. (a) $x \leq -3$ or $x > 1$; (b) $x < 0$ or $x > 3$; (c) $-1 \leq x < 2$; (d) $-1 < x < 1$



3. (a) $x < -4$ or $x > 5$; (b) $9 < x < 11$; (c) $x > 7$ or $x < 3$;
 (d) $-\frac{1}{3} < x < \frac{2}{3}$; (e) $-\frac{4}{5} < x \leq 3$; (f) $\frac{2}{2} \leq x \leq 2$; (g) $-3 \leq x < 1$;
 (h) $x < -6$ or $x > 1$; (i) $x > 3$ or $x < -1$; (j) $-\frac{1}{2} \leq x < \frac{3}{2}$;
 (k) $\frac{10}{3} \leq x \leq 5$; (l) no solution.

Exercise 3

1. $x \geq 11$ or $x \leq 3$; 2. $-3 < x < 2$; 3. $4 \leq s \leq 12$; 4. $x \geq 7$ or $x \leq -1$;
 5. no solution; 6. $x \neq 5$; 7. $-17 < x < 13$; 8. all x ;
 9. $x < -\frac{38}{11}$ or $x > 4$; 10. $x \geq \frac{3}{5}$ or $x \leq \frac{1}{25}$; 11. no solution;
 12. $x > 3$ or $x < -3$.

Exercise 4

1. $0 \leq x \leq 4$; 2. $x \geq 1$ or $x \leq -7$; 3. $1 - \sqrt{5} < x < 1 + \sqrt{5}$;
~~4. $x \geq -4$ or $x \leq 6$; 5. $-1 - \sqrt{5} \leq x \leq -1 + \sqrt{5}$; 6. $x \geq 3 + \sqrt{2}$ or $x \leq 3 - \sqrt{2}$;~~

7. $x > 2 + \sqrt{14}$ or $x < 2 - \sqrt{14}$; 8. $x > \sqrt{\frac{43}{12}} - \frac{1}{2}$ or $x < -\sqrt{\frac{43}{12}} - \frac{1}{2}$;

9. $-3 \leq x \leq 1$; 10. $x \geq \sqrt{6}$ or $x \leq -\sqrt{6}$; 11. $x = 8$;

12. $x > -2 + \frac{\sqrt{3}}{2}$ or $x < -2 - \frac{\sqrt{3}}{2}$

Exercise 5

1. $x \geq 13$ or $x \leq 0$; 2. $x \geq 4$; 3. $x < -4$ or $-1 \leq x \leq 2$; 4. $x \leq -3$ or $3 \leq x \leq 7$;

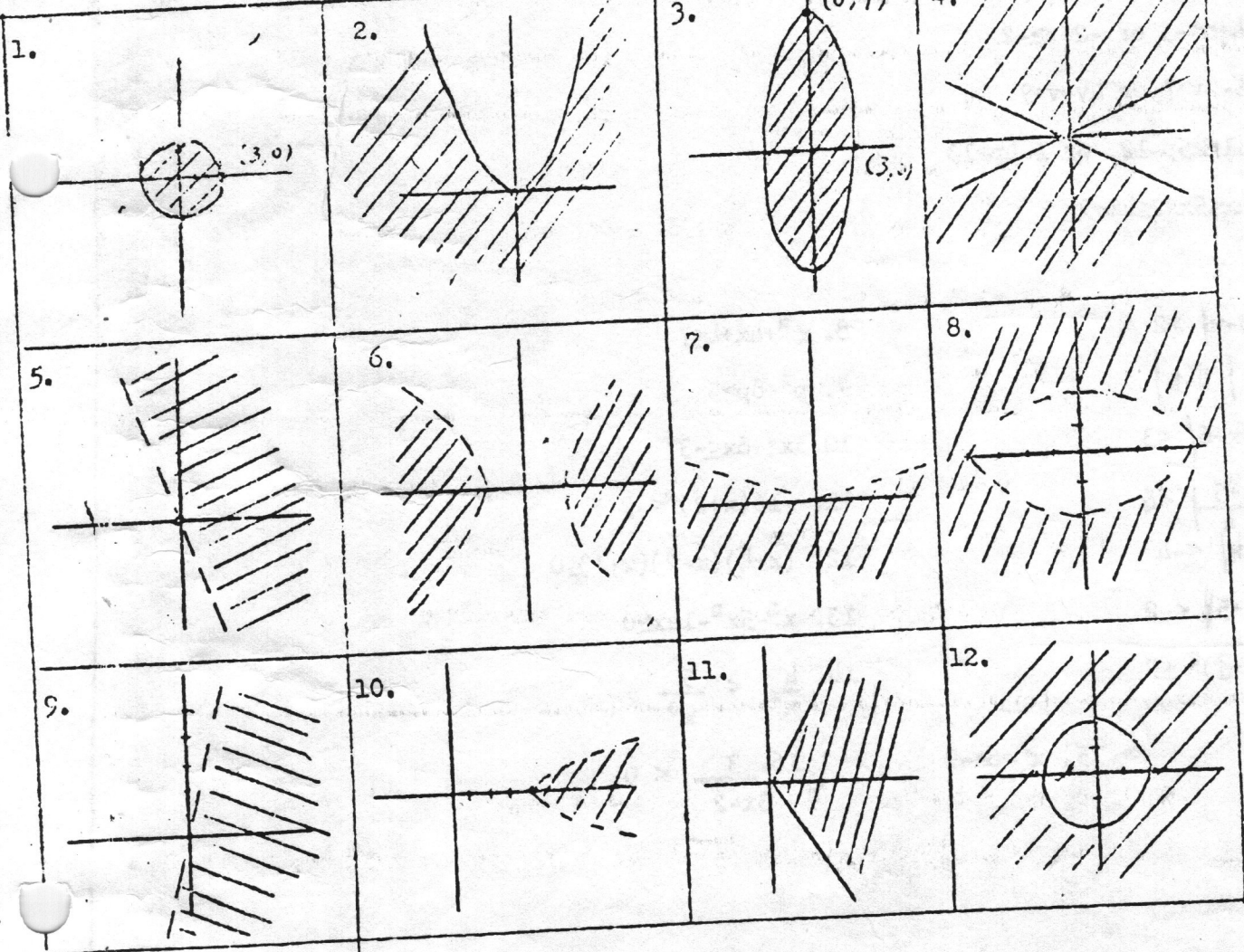
5. $-2 < x < 7$; 6. $x < -6$ or $-1 < x < 0$; 7. all x ; 8. $x < 0$ or $x > 2$;

9. $-\frac{2}{3} \leq x < \frac{7}{2}$; 10. $x < -3$ or $0 \leq x < 4$; 11. $x < -\frac{13}{2}$; 12. $1 < x < \frac{7}{2}$;

13. $x > .1$ or $x < -.1$; 14. $-10 \leq x \leq 20$; 15. $x \neq 5$; 16. $x < -3$ or $-\frac{5}{2} < x < 4$;

17. $x > \frac{7}{2}$

Exercise 6



I. Solve and Graph the solution:

1. $5a - 1 \geq 9$

2. $2 - 3s \leq 11$

3. $-14 \leq 3t - 2$

4. $3x + 5 < x - 5$

5. $-5w + 5 > -(13 + w)$

6. $-6m + 12 \leq 12 + 6m$

7. $\frac{3x+2}{3} < \frac{x+5}{6}$

8. $-2 + 11b < 7(2b - 5) - 17$

9. $-8 \leq -1 + 3a < 11$

10. $-7 \leq 4b - 5 \leq 19$

11. $4 + n \leq -3$ or $-2 + n \geq -2$

12. $5 - 2v > 7$ or $4v > v + 9$

13. $11p \geq 5p - 12$ or $1 - 4p > 13$

14. $4x \leq 5x + 2 \leq 4x + 9$

II. Solve

1. $|3 - w| > 2$

2. $1 < |4 - k|$

3. $|3x + 6| < 3$

4. $\left| \frac{x+5}{4} \right| < 6$

5. $-|w| < -4$

6. $|a + 5| < -2$

7. $(a + 8)^2 \leq 9$

8. $x^2 + 4x + 4 \leq 7$

9. $p^2 + 8p > 5$

10. $3x^2 + 6x \leq -3$

11. $-16(x + 2)^2 < -9$

12. $(x + 4)(x - 2)(x + 7) \leq 0$

13. $x^3 - 5x^2 - 14x \geq 0$

14. $\frac{4}{x^2} < \frac{5}{x^3}$

15. $x^2 > 6x - 9$

16. $\frac{3}{3x - 2} < 0$

III. Graph these:

1. $x^2 + y^2 \leq 9$

4. $4x^2 + 9y^2 < 36$

7. $x^2 - y^2 > 25$

2. $x^2 + y^2 > 25$

5. $3x + 2y < 8$

8. $y < x^2 + 3$

3. $4x^2 + y^2 \geq 16$

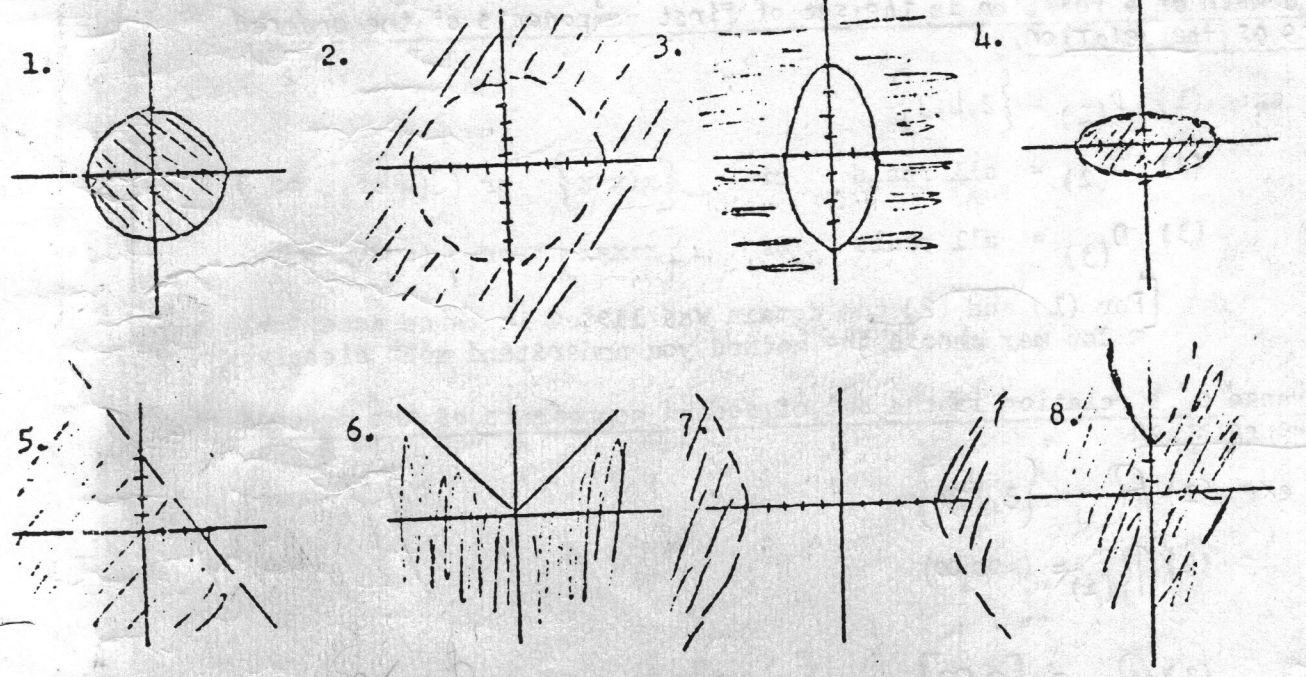
6. $y \leq |x|$

ANSWERS

- I. 1. $a > 2$
 2. $s > -3$
 3. $t \geq -4$
 4. $x < -5$
 5. $w < 9/2$
 6. $m > 0$
 7. $\bar{x} < 1/5$
 8. No solution
 9. $-7/3 \leq a < 4$
 10. $-1/2 \leq b \leq 6$
 11. $n < -7$ or $n \geq 0$
 12. $v < -1$ or $v > 3$
 13. $p < -3$ or $p \geq -2$
 14. $-2 \leq x \leq 7$

- II. 1. $w < 1$ or $w > 5$
 4. $-29 < x < 19$
 7. $-11 \leq a \leq -5$
 9. $p > -4 + \sqrt{21}$ or $p < -4 - \sqrt{21}$
 10. $x = -1$
 11. $x > -1.25$ or $x < -2.75$
 12. $x \leq -7$ or $-4 \leq x \leq 2$
 13. $-2 \leq x \leq 0$ or $x \geq 7$
 14. $0 < x < 1.25$
 15. $x \neq 3$
 16. $x < 2/3$

III.



$y = mx + 4$