ALGEBRA II
BINOMIAL EXPANSION

OBJECTIVES:

I. DEFINE
   A. $n! = n(n-1)(n-2) \ldots 3 \cdot 2 \cdot 1$ for $n \geq 1$
   B. $0! = 1! = 1$
   C. ${}_nC_r$ --- Combination

II. Be able to write a given number of lines of Pascal's Triangle.

III. Given $(a \pm b)^n$, be able to
   A. Find the number of terms in the expansion.
   B. Expand the binomial
      1. using the recursive method
      2. using the explicit method
   C. Be able to find a specific term in the expansion

I. Factorial Notation and Combinations

A. In mathematics it is often necessary to find the product of consecutive positive integers. $(5 \cdot 4 \cdot 3 \cdot 2 \cdot 1)$ This is symbolized by using an exclamation point ! --- this is called factorial notation.

example: $5! = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120$

example: $4!\,2! = (4 \cdot 3 \cdot 2 \cdot 1)(2 \cdot 1) = (24)(2) = 48$

example: $\dfrac{7!\,3!}{5!} = \dfrac{(7 \cdot 6 \cdot \cancel{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1})(3 \cdot 2 \cdot 1)}{\cancel{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}} = (42)(6) = 252$

EXERCISE I FIND THE FOLLOWING PRODUCTS

1. $4!$
2. $6!$
3. $\dfrac{9!}{5!}$
4. $\dfrac{13!\,2!}{10!}$
5. $\dfrac{5!\,7!}{8!}$
6. $\dfrac{(8-2)!}{(4+1)!}$
7. $\dfrac{7!\,4!}{6!}$
8. $\frac{1}{2}(6!)$
9. $\dfrac{10!}{8!}$
10. $0!\,4!$
11. $\dfrac{x!}{(x-2)!}$
12. $\dfrac{(x+1)!}{(x-1)!}$
13. $\dfrac{(n+2)!}{n!}$
14. $\dfrac{x!\,(x-3)!}{(x-2)!\,(x-1)!}$

B. Factorial notation is used when trying to combine a certain number of elements in a set without regard to a specific order.. This combination is symbolised by either ${}_nC_r$ or by $\binom{n}{r}$. To find the number of possible combination, the following formula is used.

$$\binom{n}{r} = {}_nC_r = \frac{n!}{r!\,(n-r)!}$$

example: ${}_5C_2 = \frac{5!}{2!\,(5-2)!} = \frac{5\cdot4\cdot3\cdot2\cdot1}{2\cdot1\,(3\cdot2\cdot1)} = 10$

example: $\binom{7}{3} = \frac{7!}{3!\,(7-3)!} = \frac{7\cdot6\cdot5\cdot4\cdot3\cdot2\cdot1}{3\cdot2\cdot1\,4\cdot3\cdot2\cdot1} = 35$

example: ${}_{n+1}C_n = \frac{(n+1)!}{n!\,(n+1-n)!} = \frac{(n+1)(n)!}{n!\,(1)!} = n+1$

EXERCISE II Find the following combinations

1. $\binom{4}{1}$
2. $\binom{5}{3}$
3. ${}_7C_4$
4. ${}_8C_3$
5. $\binom{8}{8}$
6. ${}_{12}C_3$
7. ${}_{30}C_{29}$
8. ${}_{15}C_{10}$
9. $\binom{n+2}{n}$
10. $\binom{n+2}{n-1}$
11. $\binom{5}{8}$
12. ${}_8C_0$

II. Examine the following expansions and then answer the questions by refering to the examples. The methods of producing the expansions are not shown but can be verified by multiplication.

$(x + y)^0 = 1$

$(x + y)^1 = 1x + 1y$

$(x + y)^2 = 1x^2 + 2xy + 1y^2$

$(x + y)^3 = 1x^3 + 3x^2y + 3xy^2 + 1y^3$

$(x + y)^4 = 1x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + 1y^4$

$(x - y)^0 = 1$

$(x - y)^1 = 1x - 1y$

$(x - y)^2 = 1x^2 - 2xy + 1y^2$

$(x - y)^3 = 1x^3 - 3x^2y + 3xy^2 - 1y^3$

$(x - y)^4 = 1x^4 - 4x^3y + 6x^2y^2 - 4xy^3 + 1y^4$

EXERCISE III. In the expansion of $(x \pm y)^n$

1. The number of terms of $(x \pm y)^n$ is ________.
2. The coefficient of the first and last term is always ________.
3. The exponent of x in any term after the first is ______ ______ than the exponent of x in the preceding term.
4. The exponent of y in any term is ______ ______ than the exponent of y in the preceding term.
5. The sum of the exponents of x and y in <u>each</u> term is ______.
6. (x - y) is treated as (x + ___).

Now in trying to expand any binomial without multiplying it out we need two things. 1) a pattern for the variables and 2) a method for determining the coefficients of each term. A closer examination of the exponents in the examples will reveal that any expansion will follow this pattern.

$(x + y)^7 = \_x^7 + \_x^6y + \_x^5y^2 + \_x^4y^3 + \_x^3y^4 + \_x^2y^5 + \_xy^6 + \_y^7$

$(x - y)^4 = \_x^4 - \_x^3y + \_x^2y^2 - \_xy^3 + \_y^4$

To determine the coefficients there are three methods. The first of which is called Pascal's Triangle.

$(a + b)^0 =$ 1

$(a + b)^1 =$ 1 1

$(a + b)^2 =$ 1 2 1

$(a + b)^3 =$ 1 3 3 1

$(a + b)^4 =$ 1 4 6 4 1

OBSERVE: The 1st and the last term of each row is 1. Each other term is the sum of the two terms to the right and left of it in the preceding row.

So to write any expansion simply write the variable part first and then select the coefficients from the appropriate line of Pascal's Triangle.

EXERCISE IV Use Pascal's Triangle down to $n = 7$ to find the numerical coefficier coefficients of the following and then wirte the ex ansion.

1. $(x + y)^5$
2. $(a + b)^6$
3. $(r - s)^5$
4. $(x - 2)^7$
5. $(x + 2y)^4$
6. $(x - y)^3$

B. Another method of determining the numerical coefficients is the recursive method. Here we first write the expansion and then find the coefficients by multiplying the exponent of the preceding term by the coefficient of the preceding term and then dividing by the number of the preceding term (here we begin with the second term as the coefficient of the first term is one). Study the following exapmle closely.

$$(a + b)^5 = 1a^5 + \frac{5\cdot 1}{1}a^4b + \frac{4\cdot 5}{2}a^3b^2 + \frac{10\cdot 3}{3}a^2b^3 + \frac{10\cdot 2}{4}ab^4 + \frac{5\cdot 4}{5}b^5$$

$$= 1a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + 1b^5$$

EXERCISE V. Expand using the recursive method

1. $(a+b)^2$
2. $(a + b)^4$
3. $(2 - c)^5$
4. $(x - 5)^3$
5. $(2x - 3)^4$
6. $(r + 4s)^4$
7. $(x + 1)^5$
8. $(5 - x)^3$

C. The third method of determining the numerical coefficients is called the explicit method. Here we shall utilize the combination formula learned in Section I part B. Review that section and then examine the following example.

$$(a + b)^4 = \binom{4}{0}a^4 + \binom{4}{1}a^3b + \binom{4}{2}a^2b^2 + \binom{4}{3}ab^3 + \binom{4}{4}b^4$$

$$= \frac{4!}{0!\,4!}a^4 + \frac{4!}{1!\,3!}a^3b + \frac{4!}{2!\,2!}a^2b^2 + \frac{4!}{3!\,1!}ab^3 + \frac{4!}{4!\,0!}b^4$$

$$= 1a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + 1b^4$$

EXERCISE VI. Expand using the explicit method.

1. $(x + y)^6$
2. $(a - b)^5$
3. $(x + 2y)^3$
4. $(x - 3)^4$
5. $(2x + 5)^4$
6. $(x - y)^6$

III. Finding the specific term

We can use the explicit method of determining the coefficients when trying to find a specific term in an expansion without expanding the binomial. The following formula gives us a means of finding any specific term.

$$\binom{n}{r} x^{n-r} y^r$$

example Find the 7th term of $(x + y)^{10}$.

$$\binom{10}{6} x^{10-6} y^6 = \frac{10!}{6!\,4!} x^4y^6 = 210\, x^4y^6$$

*********NOTE: r = 1 less than the term you are looking for
n is the power of the expansion
the sum of the powers on x and y equals n

example: Find the 4th term of $(x - 2y)^9$.

$$\binom{9}{3} x^{9-3} (-2y)^3 = \frac{9!}{3!6!} x^6 (-2y)^3$$

$$= \frac{9 \cdot 8 \cdot 7 \cdot 6!}{3 \cdot 2 \cdot 6!} x^6 (-8y^3)$$

$$= (84)(x^6)(-8y^3)$$

$$= -672\, x^6y^3$$

EXERCISE VII. Find the following specific terms

1. $(x + y)^6$; 3rd term
2. $(x + y)^8$; 6th term
3. $(x + y)^9$; 4th term
4. $(x - y)^7$; 5th term
5. $(x - 2y)^6$; 4th term
6. $(x + y)^6$; 4th term
7. $(x + y)^8$; 3rd term
8. $(x + y)^9$; 5th term
9. $(x - y)^6$; 4 th term
10. $(x - 2y)^7$; 6 th term

NOW ON TO THE TRIAL RUN !!!!!!!!

EXERCISE I

1. 24
2. 720
3. 3,024
4. 3432
5. 15
6. 6
7. 168
8. 360
9. 90
10. 24
11. $x^2 - x$
12. $x^2 + x$
13. $n^2 + 3n + 2$
14. $\frac{x}{x - 2}$

EXERCISE II

1. 4
2. 10
3. 35
4. 56
5. 1
6. 220
7. 30
8. 3003
9. $\frac{(n + 2)(n + 1)}{2}$
10. $\frac{(n + 2)(n + 1)(n)}{6}$
11. no solution
12. 1

EXERCISE III

1. $n + 1$
2. one
3. one less
4. one more
5. n
6. $(-y)$

EXERCISE IV

1. $x^5 + 5x^4y + 10x^3y^2 + 10x^2y^3 + 5xy^4 + y^5$
2. $a^6 + 6a^5b + 15a^4b^2 + 20a^3b^3 + 15a^2b^4 + 6ab^5 + b^6$
3. $r^5 - 5r^4s + 10r^3s^2 - 10r^2s^3 + 5rs^4 - s^5$
4. $x^7 - 14x^6 + 84x^5 - 280x^4 + 560x^3 - 672x^2 + 448x - 128$
5. $x^4 + 8x^3y + 24x^2y^2 + 32xy^3 + 16y^4$
6. $x^3 - 3x^2y + 3xy^2 - y^3$

EXERCISE V

1. $a^2 + 2ab + b^2$
2. $a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$
3. $32 - 80c + 80c^2 - 40c^3 + 10c^4 - c^5$
4. $x^3 - 15x^2 + 75x - 125$
5. $16x^4 - 96x^3 + 216x^2 - 216x + 81$
6. $r^4 + 16r^3s + 96r^2s^2 + 256rs^3 + 256s^4$
7. $x^5 + 5x^4 + 10x^3 + 10x^2 + 5x + 1$
8. $125 - 75x + 15x^2 - x^3$

EXERCISE VI

1. $x^6 + 6x^5y + 15x^4y^2 + 20x^3y^3 + 15x^2y^4 + 6xy^5 + y^6$

2. $a^5 - 5a^4b + 10a^3b^2 - 10a^2b^3 + 5ab^4 - b^5$

3. $x^3 + 6x^2y + 12xy^2 + 8y^3$

4. $x^4 - 12x^3 + 54x^2 - 108x + 81$

5. $16x^4 + 160x^3 + 600x^2 + 1000x + 625$

6. $x^6 - 6x^5y + 15x^4y^2 - 20x^3y^3 + 15x^2y^4 - 6xy^5 + y^6$

EXERCISE VII

1. $15x^4y^2$
2. $56x^3y^5$
3. $84x^6y^3$
4. $35x^3y^4$
5. $-160x^3y^3$
6. $20x^3y^3$
7. $28x^6y^2$
8. $126x^5y^4$
9. $-20x^3y^3$
10. $-672x^2y^5$

TRIAL RUN — BINOMIAL EXPANSION — ALGEBRA II

I. SIMPLIFY

1. $5!$
2. $0! = 1$
3. $\frac{1}{2}(4!)$
4. $3!\,4! = 144$
5. $\frac{10!}{8!}$
6. $\frac{x!}{(x-2)!}$
7. $\frac{(x+1)!}{(x-1)!}$
8. $\frac{x!\,(x-3)!}{(x-2)!\,(x-1)!}$

II, SIMPLIFY

1. $\binom{4}{2}$
2. $\binom{7}{7}$
3. ${}_8C_5$
4. $\binom{n+1}{n}$
5. $\binom{20}{0}$
6. ${}_{15}C_{10}$
7. $\binom{n+2}{n-1}$
8. $\binom{30}{29}$
9. $\binom{5}{8}$

III. Use Pascal's Triangle to expand $(a + b)^6$.

IV. Expand each of the following using the ~~recursive~~ pascal method.

1. $(a + b)^4$
2. $(2 - x)^6$
3. $(a + 5)^8$
4. $(2a - 3b)^5$

V. Expand each of the following using the Explicit method.

1. $(a + b)^5$
2. $(4 - s)^4$
3. $(2x+3y)^9$
4. $(3r + 2s)^4$

VI. Find the specific term in the expansion of each of the following binomials:

1. 5th, $(a + b)^{10}$
2. 4th, $(r - 2s)^7$
3. 8th, $(a + 3b)^8$
4. 5th, $(2a - b)^9$

VII. Write the first seven rows of Pascal's Triangle.

TRIAL RUN ANSWERS

BINOMIAL EXPANSION

I.
1. 120
2. 1
3. 12
4. 144
5. 90
6. $x(x - 1)$
7. $x(x + 1)$
8. $\frac{x}{x - 2}$

II.
1. 6
2. 1
3. 56
4. $n + 1$
5. 1
6. 3,003
7. $\frac{(n + 2)(n + 1)(n)}{6}$
8. 30
9. no solution

III. $a^6 + 6a^5b + 15a^4b^2 + 20a^3b^3 + 15a^2b^4 + 6ab^5 + b^6$

IV.
1. $a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$
2. $64 - 192x + 240x^2 - 160x^3 + 60x^4 - 12x^5 + x^6$
3. $a^8 + 40a^7 + 28a^6(5)^2 + 56a^5(5)3 + 70a^4(5)^4 + 56a^3(5)^5 + 28a^2(5)^6 + 8a(5)^7 + (5)^8$
4. $(2a)^5 - 5(2a)^4(3b) + 10(2a)^3(3b)^2 - 10(2a)^2(3b)^3 + 5(2a)(3b)^4 - (3b)^5$

V.
1. $a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5$
2. $256 - 256s + 96s^2 - 16s^3 + s^4$
3. $x^9 - 9x^8y + 36x^7y^2 - 84x^6y^3 + 126x^5y^4 - 126x^4y^5 + 84x^3y^6 - 36x^2y^7 + 9xy^8 - y^9$
4. $(3r)^4 + 4(3r)^3(2s) + 6(3r)^2(2s)^2 + 4(3r)(2s)^3 + (2s)^4$

VI.
1. $210a^6b^4$
2. $-280r^4S^3$
3. $8a(3b)^7$
4. $126(2a)^5b^4$

VII.

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1 1
1 6 15 20 15 6 1
1 7 21 35 35 21 7 1