EQUATIONS

BEHAVIORAL OBJECTIVES

I. Be able to solve, over the set of real numbers, equations involving
   - A. Fractions
   - B. Quadratics
   - C. Polynomials of degree greater than two
   - D. Radicals
   - E. Absolute value
   - F. The variable as an exponent

II. Given formulas commonly used in the applications of mathematics, solve for a given variable

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SECTION I — EQUATIONS INVOLVING FRACTIONS AND QUADRATICS

Probably one of the most common tasks a mathematician is called upon to do is to solve an equation of some nature. It is an important skill and the better skilled an individual becomes, the freer he is to charge ahead in the field of mathematics. This L.A.P. explores some types of equations and some techniques used to solve them.

Some basic principles:

1. Given an equation, adding or subtracting the same number from both sides of the equation preserves the original equation.
2. Given an equation, multiplying or dividing both sides of the equation by the same number, as long as that number does not contain the variable, preserves the equation. UNLESS THAT NUMBER IS ZERO! When multiplying both sides by an expression which contains the variable, extranious roots might occur. These are later rejected because they will not check. Dividing both sides by an expression which contains a variable might cause the loss of a root. Please avoid losing roots!
3. If a product is zero, then each factor may be set equal to zero.

Since this section deals with quadratic equations; $ax^2 + bx + c = 0$ for $a \neq 0$, recall that some quadratics may be easily factored. All quadratic equations can be solved by using the quadratic formula.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

EXAMPLE 1: Solve the following equation for x:

$$\frac{200}{x} - 1 = \frac{200}{x + 10}$$

$(x + 10)200 - x(x + 10) = 200x$	(Multiply both sides by $x(x + 10)$)
$200x + 2000 - x^2 - 10x = 200x$	(Expand)
$-x^2 - 10x + 2000 = 0$	(Combine like terms. Equate to zero.)
$x^2 + 10x - 2000 = 0$	(Multiply both sides by -1)
$(x - 40)(x + 50) = 0$	(Factor)
$x = 40, \quad x = -50$	(Set each factor equal to zero and solve.)

The solution set for this equation is $\{40, -50\}$. Both solutions check in the original equation.

EXAM[illegible] Solve the following equation for x:

$$\frac{2}{x} - 2 = \frac{-1}{x^2}$$

$2x - 2x^2 = -1$ (Multiply by x^2)

$2x - 2x^2 + 1 = 0$ (Equate to zero)

$2x^2 - 2x - 1 = 0$ (Multiply by -1, arrange in descending powers of x)

$$x = \frac{2 \pm \sqrt{4 + 8}}{4}$$

$$x = \frac{2 \pm 2\sqrt{3}}{4}$$

$$x = \frac{1 \pm \sqrt{3}}{2}$$

The solution set is: $\left\{\frac{1 + \sqrt{3}}{2}, \frac{1 - \sqrt{3}}{2}\right\}$

EXERCISE 1 Solve each of the following for x: (Give only real number solutions.)

1. $\frac{4}{x} - \frac{1}{2} = \frac{5}{12} - \frac{3}{2x}$
2. $\frac{10}{1 - 2x} = 2$
3. $\frac{3}{5 - 3x} = \frac{1}{2}$
4. $\frac{x + 1}{x - 3} - \frac{2x - 4}{x^2 - 9} = 1$
5. $\frac{3}{x^2 - x} = \frac{3}{x - 1} - \frac{5}{x}$
6. $\frac{x}{2} + \frac{x}{3} + \frac{x}{4} = 26$
7. $5x^2 + 2 = 11x$
8. $2x^2 - 5x - 4 = 0$
9. $\frac{3}{x^2} - \frac{5}{x} + 2 = 0$
10. $\frac{2}{(x + 1)^2} - \frac{5}{x + 1} = -2$

SECTION II EQUATIONS OF DEGREE HIGHER THAN TWO

Not all equations of degree greater than two succumb to factoring. However, those which do cooperate and factor nicely, can be solved with ease.

EXAMPLE 1: Solve for all real number values of x:

$$x^3 - 2x^2 + x = 0$$

$$x(x^2 - 2x + 1) = 0$$

$$x(x - 1)^2 = 0$$

The solution set is $\{0, 1\}$

EXAMPLE 2: $x^3 - 8 = 0$

$(x - 2)(x^2 + 2x + 4) = 0$

$x = 2$ is the only real number solution.

For the quadratic part, notice that the value of the discriminant is less than zero.

EXAMPLE 3: $x^4 - 16 = 0$

$(x^2 + 4)(x - 2)(x + 2) = 0$

The real number solution set is $\{2, -2\}$

EXAMPLE 4: $x^5 - 3x^4 - 4x^3 + 12x^2 + 4x - 12 = 0$

$x^4(x - 3) - 4x^2(x - 3) + 4(x - 3) = 0$

$(x - 3)(x^4 - 4x^2 + 4) = 0$

$(x - 3)(x^2 - 2)(x^2 - 2) = 0$

The solution set is: $\{3, \sqrt{2}, -\sqrt{2}\}$

EXERCISE 2 Give all real number solutions for the following equations.

1. $x^4 - 6x^2 + 9 = 0$
2. $x^3 + 2x^2 - 35x = 0$
3. $x^4 + 8x^3 - x^2 - 8x = 0$
4. $x^8 - 256 = 0$
5. $(x - 10)(x^2 + 11x + 30) = 0$
6. $x^4 - 5x^2 + 6 = 0$
7. $(x + 2)(x - 1)(x^2 - 20x + 99) = 0$
8. $x^3 + 2x^2 + x = x^2 + 7x$
9. $x^3 + 3x^2 - x - 3 = 0$
10. $(x + 8)(x^2 + 3x + 1) = 0$

SECTION III EQUATIONS INVOLVING RADICAL EXPRESSIONS

If $\sqrt{x} = 7$ then it is relatively obvious that x must be 49. What we are actually doing is squaring both sides of the equation. Recall, so long as we do the same operation to both sides of an equation, the original equation is preserved. At times we may introduce extranious roots; i.e. values for x which will not satisfy the original equation. This we discover when we check our solutions. Dividing both sides of an equation may cause us to loose roots. This is not so good! Of course too, multiplying or dividing by zero is out! Raising both sides of an equation to the same power is a fair enough operation.

EXAMPLE 1: number values of x:

$\sqrt{5x+6} - \sqrt{x+3} = 3$

$\sqrt{5x+6} = 3 + \sqrt{x+3}$ (Isolate one radical)

$5x + 6 = 9 + 6\sqrt{x+3} + x + 3$ (Square both sides)

$4x - 6 = 6\sqrt{x+3}$ (Combine like terms and isolate the radical.)

$2x - 3 = 3\sqrt{x+3}$ (Divide by 2 to simplify)

$4x^2 - 12x + 9 = 9(x+3)$ (Suuare both sides again.)

$4x^2 - 21x - 18 = 0$ (Combine like terms and equate to zero.)

$(4x+3)(x-6) = 0$ (Factor)

$x = -\frac{3}{4}, \quad x = 6$

The solution set is $\{6\}$. Notice, $x = -\frac{3}{4}$ does not check in the original.

EXAMPLE 2:

$\sqrt[4]{4 - 11x} = 3$

$4 - 11x = 81$ (Raise both sides to the 4th power.)

$-11x = 77$

$x = -7$

The solution set is $\{-7\}$.

EXAMPLE 3:

$x - 3\sqrt{x} - 28 = 0$

$(\sqrt{x} - 7)(\sqrt{x} + 4) = 0$

$\sqrt{x} = 7; \quad \sqrt{x} = -4$ (reject. $\sqrt{x}$ must be ≥ 0)

$x = 49$

The solution set is $\{49\}$

EXAMPLE 4:

$x^{1/3} + x^{1/6} - 2 = 0$

$(x^{1/6} - 1)(x^{1/6} + 2) = 0$

$x^{1/6} = 1 \quad x^{1/6} = -2$ (Reject. An even root must always be ≥ 0.)

The solution set is $\{1\}$.

EXERCISE 3 Solve each of the following equations for all real number values of x:

1. $\sqrt{4x + 9} = 7$

2. $5 - \sqrt[3]{3x^2 + 2} = 0$

3. $\sqrt{5x - 4} - 2 = 0$

4. $2x - \sqrt{2x - 3} = 3$

5. $(\sqrt{x} - 1)^2 - 6(\sqrt{x} - 1) + 9 = 0$

6. $x - 7\sqrt{x} + 12 = 0$

7. $\sqrt{3x + 4} = -8$

8. $\sqrt{2x - 3} - \sqrt{x + 7} + 2 = 0$

9. $\sqrt[3]{x^2 - 1} = 2$

10. $x^{2/3} + x^{1/3} - 6 = 0$

11. $2x^{2/3} - x^{1/3} - 6 = 0$

12. $x + 2\sqrt{x} - 15 = 0$

13. $\sqrt{x^2 + 5} = (-3)^2$

14. $\dfrac{x}{\sqrt{x + 3}} = 2$

SECTION 4 EQUATIONS INVOLVING ABSOLUTE VALUE

Absolute value is an operation which makes any non-zero number a positive number.

$$|a| \geq 0, \quad |0| = 0$$

For all real numbers x, $\sqrt{x^2} = |x|$

Examples: $|8| = 8$; $|-8| = 8$.

If $|x| = 4$, then $x = 4$ or $x = -4$

Keeping the above information in mind, study the following examples:

EXAMPLE 1: Solve for x:

$$|2x + 1| = 7$$

$2x + 1 = 7$ or $2x + 1 = -7$

$2x = 6$ or $2x = -8$

$x = 3$ or $x = -4$

The solution set is 3, -4

EXAMPLE 2: Solve for x:

$$|2x + 1| = |x|$$

The above expression is equivalent to: $\sqrt{(2x + 1)^2} = \sqrt{x^2}$

So: $(2x + 1)^2 = x^2$

$4x^2 + 4x + 1 = x^2$

$3x^2 + 4x + 1 = 0$

$(3x + 1)(x + 1) = 0$

$x = -1/3$ or $x = -1$

The solution set is $\{-1/3, -1\}$

EXERCISE 4 Solve each of the following equations for all real number values of x.

1. $|25x - 8| = 7$
2. $|7 - 2x| = 1$
3. $|5x - 2| = |2x + 13|$
4. $|x - 7| = |3x + 1|$
5. $|x| = |-3|$
6. $3|x| = 15$
7. $\left|\frac{1}{x}\right| = |x|$
8. $|x - 3| + |x - 5| = 8$
9. $|x - 1| - 2|x - 3| = 4$
10. $\left|\frac{2}{x} - 3\right| = 1$

SECTION 5 EQUATIONS INVOLVING THE VARIABLE AS AN EXPONENT

Before actually giving some examples of equations to solve involving the variable as an exponent, we shall review the basic rules regarding exponents.

1) $x^a x^b = x^{a+b}$
2) $x^a \div x^b = x^{a-b}$
3) $(x^a)^b = x^{ab}$
4) $x^{\frac{a}{b}} = \sqrt[b]{x^a}$
5) $x^{-a} = \frac{1}{x^a}$
6. $\log_b a = x$ if and only if $b^x = a$

Example 1: Suppose we wich to determine the value of x for the following equation:

$$27^x = 81$$

$$(3^3)^x = 3^4 \quad \text{(Change both sides to powers of 3)}$$

$$3^{3x} = 3^4$$

$$3x = 4 \quad \text{(Since the bases are the same, equate the exponents.)}$$

$$x = \frac{4}{3}$$

Example 2: Solve the following equation for x:

$$2^{2x+2} - 9(2^x) + 2 = 0$$

$$(2^x)^2 \cdot 2^2 - 9(2^x) + 2 = 0 \quad \text{(Use the rules for exponents and change to the quadratic form.)}$$

$$4(2^x)^2 - 9(2^x) + 2 = 0$$

$$(4(2^x) - 1)(2^x - 2) = 0 \quad \text{(Factor)}$$

$$4(2^x) = 1 \qquad 2^x = 2 \quad \text{(Set each factor equal to zero and solve for x.)}$$

$$2^x = \frac{1}{4} \qquad x = 1$$

$$x = -2$$

When the bases cannot be resolved to the same number we need to resort to logs.

Example 3: Solve for x:

$$7^x = 5$$

$$\log 7^x = \log 5$$

$$x \log 7 = \log 5$$

$$x = \frac{\log 5}{\log 7}$$ (This is a good answer.)

$$x = .8$$ (With the help of a calculator the above is determined to be approximately .8.)

EXERCISE 5 Solve each of the following for x:

1. $(3^{4x-3})^{-2} = (27)^{-x-8}$
2. $2^{3-x}(4^{2x-1}) = 16$
3. $(5^{2x})(25) = (125)^{x-1}$
4. $(5^3)^{2x-6} = (5^{-2})^{4-x}$
5. $2^{x+4} = \frac{1}{8}$
6. $2^{2x+2} + 2^{x+2} = 3$
7. $9^{3x} = 4$
8. $9^{\frac{1}{x}} = 27$
9. $3^{4x} = 5$
10. $5^x - 5^{x-2} = 120\sqrt{5}$

SECTION 6 FORMULAS

Often mathematics is applied to the real world through the use of formulas. To evaluate a formula means to find the value of one letter which appears in the formula when the values of the remaining letters are known. In this section we shall not deal with actual values for the various letters in the formula, but rather simply solve the formula for a given letter in terms of the other letters of the formula.

Example: 1 $P = 2\ell + 2w$. This is the formula for finding the perimeter of a a rectangle when given the length and the width.

Suppose we knew P and w and wish to find ℓ.

$P - 2w = 2\ell$ (Subtract 2w from each side of the equation.)

$\frac{P - 2w}{2} = \ell$ (Divide both sides of the equation by 2.)

Example 2: $\frac{1}{R} = \frac{1}{r_1} + \frac{1}{r_2}$ — For parallel circuits: The reciprocal of the total resistance is the sum of the reciprocals of the several resistances.

To solve for r_2

$\frac{1}{R} = \frac{r_2 + r_1}{r_1 r_2}$ — Combine right side to a single fraction.

$r_1 r_2 = Rr_2 + Rr_1$ — Cross multiply

$r_1 r_2 - Rr_2 = Rr_1$ — Put all terms containing the to be solved for variable on one side of the equation.

$r_2(r_1 - R) = Rr_1$ — Factor

$r_2 = \frac{Rr_1}{r_1 - R}$ — Divide by $(r_1 - R)$

EXERCISE 6 — For each of the following standard formulas, solve for the indicated variable.

1. $V = \frac{4}{3}\pi r^3$; Solve for r; (Volume of a sphere= $\frac{4}{3}\pi$ (radius)^3).

2. $A = \frac{h(B + b)}{2}$; Solve for B; (Area of a trapezoid = $\frac{\text{height(sum of bases)}}{2}$)

3. $V = \frac{1}{3}\pi r^2 h$; Solve for h; (Volume of a right circlular cone = $\frac{1}{3}\pi$ (hight)(radius)^2)

4. $A = \frac{s^2\sqrt{3}}{4}$; Solve for s; (Area of an equilateral triangle = $\frac{(\text{side})^2\sqrt{3}}{4}$)

5. $C = \frac{5}{9}(F - 32)$; Solve for F; (Centigrade = $\frac{5}{9}$(Fahrenheit - 32))

6. $F = \frac{mv^2}{r}$; Solve for v; (Centripetal force = $\frac{(\text{mass})(\text{velocity})^2}{\text{radius}}$)

7. $f = \frac{1}{2\pi\sqrt{LC}}$; Solve for C; (Frequency = $\frac{1}{2\pi\sqrt{(\text{Inductance})(\text{Capacity})}}$)

SECTION 7 — EVALUATION

1. Review the objectives of this L.A.P.
2. Take the trial run.
3. Take the test on this L.A.P.

EQUATIONS L.A.P. ANSWERS

Exercise 1: 1. 6; 2. -2; 3. $-\frac{1}{3}$; 4. -8; 5. No solution; 6. 24;

7. 2, $\frac{1}{5}$; 8. $\frac{5 \pm \sqrt{57}}{4}$; 9. $\frac{3}{2}$, 1; 10. $-\frac{1}{2}$, 1

Exercise 2: 1. $\pm\sqrt{3}$; 2. 0, 5, -7; 3. 0, -8, $\pm$ 1; 4. $\pm$ 2;

5. 10, -5, -6; 6. $\pm\sqrt{3}$, $\pm\sqrt{2}$; 7. -2, 1, 11, 9; 8. 0, -3, 2;

9. -3, $\pm$ 1; 10. -8, $\frac{-3 \pm \sqrt{5}}{2}$

Exercise 3: 1. 10; 2. $\pm\sqrt{41}$; 3. $\frac{8}{5}$; 4. $\frac{3}{2}$, 2; 5. 16;

6. 9, 16; 7. No solution; 8. 2; 9. $\pm$ 3; 10. 8, -27;

11. $-\frac{27}{-8}$, 8; 12. 9; 13. $\pm 2\sqrt{19}$; 14. 6

Exercise 4: 1. $\frac{3}{5}$, $\frac{1}{25}$; 2. 4, 3; 3. $\frac{-11}{7}$, 5; 4. $\frac{3}{2}$, -4; 5. $\pm$ 3;

6. $\pm$ 5; 7. $\pm$ 1; 8. 0, 8; 9. No solution; 10. $\frac{1}{2}$, 1

Exercise 5: 1. 6; 2. 1; 3. 5; 4. $\frac{5}{2}$; 5. -7;

6. Solution:

$x = -1$

7. $\frac{\log 2}{3 \log 3}$; 8. $\frac{2}{3}$ 9. $\frac{\log 5}{4 \log 3}$; 10. $\frac{7}{2}$

Exercise 6: 1. $\sqrt[3]{\frac{3V}{4\pi}}$; 2. $\frac{2A}{h} - b$; 3. $\frac{3V}{\pi r^2}$; 4. $\sqrt{\frac{4A}{\sqrt{3}}}$

5. $\frac{9C}{5} + 32$; 6. $\sqrt{\frac{Fr}{m}}$; 7. $\frac{1}{4 L \pi^2 f^2}$

LGEBRA 2 EQUATIONS TRIAL RUN

olve each of the following equations. Give all real values.

1. $7(2 - x) + 3(x + 1) = 8$

2. $2\sqrt{x + 4} - x = 1$

3. $\frac{7}{x - 1} - \frac{6}{x^2 - 1} = 5$

4. $\frac{x}{x + 2} - \frac{4}{x + 1} = \frac{-2}{x + 2}$

5. $x^2 + 7x - 5 = 0$

6. $|x + 8| = 10$

7 $3^{4x+1} + 3^3 = 108$

8. $x^3 - 3x^2 = 10x$

9. $4^x = 5$

10. $A = 2\pi r^2 + 2\pi rh$ (Solve for h)

11. $x(4x - 1)(x + 8) = 0$

12. $\frac{1}{x} + \frac{6}{x + 4} = 1$

13. $\sqrt{x + 13} - \sqrt{7 - x} = 2$

14. $16x^2 - 8x + 1 = 0$

15. $2^{x+1} - 4 = 28$

16. $|x + 1| = |2x - 1|$

17. $3x - \frac{2}{x} = 7$

18. $3^{\frac{x}{2}} = 27$

19. $S = \frac{n(f + c)}{2}$; (Solve for f)

20. $2^{2x} - 3(2^x) = -2$

Fold back--Don't peek too early.

NSWERS

1. $\frac{9}{4}$

2. 5;

3. $2, -\frac{3}{5}$;

4. 3

5. $\frac{-7 \pm \sqrt{69}}{2}$

6. 2, -18

7. $\frac{3}{4}$

8. 0, -2, 5

9. $\frac{\log 5}{\log 4}$

10. $\frac{A - 2\pi r^2}{2\pi r}$

11. $0, \frac{1}{4}, -8$

12. -1, 4

13. 3

14. $\frac{1}{4}$

15. 4

16. 0, 2

17. $\frac{7 \pm \sqrt{73}}{6}$

18. 6

19. $\frac{2S - nc}{n}$

20. 0, 1