ALGEBRA 2 - LEARNING ACTIVITIES PACKAGE - FUNCTIONS

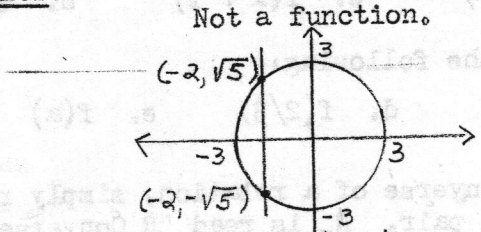
I. **Definition:** A function is a relation such that no two ordered pairs have the same first component.

Definition: The domain of a function is the set of first components of the ordered pairs of the function.

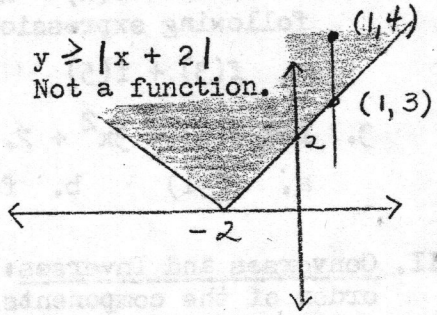
Definition: The range of a function is the set of second components of the ordered pairs of the function.

Graphically, when we select a value on the horizontal axis which belongs to the domain, there is exactly one point above or below this domain value which belongs to the function. For any given graph of a relation, consider all **vertical** lines that could be drawn across the picture. If any vertical line touches more than one point of the graph, the graph cannot represent a function. This is called the vertical line test.

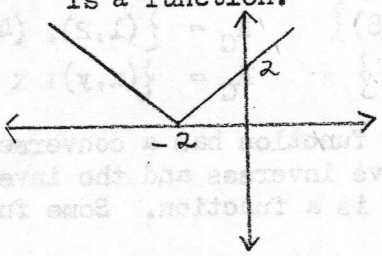
Examples: 1. $x^2 + y^2 = 9$
Not a function.



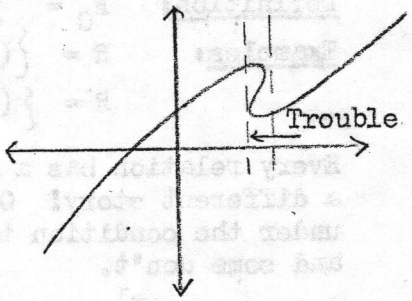
2. $y \geq |x + 2|$
Not a function.



3. $y = |x + 2|$
Is a function.



4. Not a function.



Hints: No inequality can be a function.
No equation containing a y^2 can be a function.
To use the vertical line test it isn't necessary to actually draw the lines, simply visualize them. Look for trouble spots. If none, shout "EUREKA! I'VE FOUND A FUNCTION!"

Assignment # 1: In each of the following, indicate whether or not the relation is a function. Find the domain and range of each function and graph the relation.

a. $\{(1,2), (2,3), (3,4), (5,6), (4,4)\}$ b. $y = 3$ c. $y = 3x$

d. $\{(-8,0), (-7,1), (-6,2), (-5,3), (-4,0)\}$ e. $\{(1,1), (2,1), (3,1)\}$

f. $\{(x,y): y = 5x + 1\}$ g. $|y| = |x|$ h. $y = \sqrt{x}$

i. $\{(x,y): y > x\}$ j. $x = 1$

k. $y = 3x, x \in \{-1,0,2,3\}$ l. $\{(x,y): -3x^2 = y\}$

m. $y = x^3$ n. $y = \sqrt{x - 1}$ o. $y = \frac{1}{x^2}$

p. $\{(x,y): y \leq x + 3\}$ q. $y = 25 - x^2$

II. Functional Notation: The following are commonly used notations for functions:

1. The set way: $\{(x,y): y = 1\}$
2. The functional way: $F(x) = x + 1$. (you can think $y = x + 1$)
3. The mapping way: $F: x \longrightarrow x + 1$

Assignment # 2:

1. Assume that $g(t) = t^2 + 1$. Determine each of the following expressions:

   a. $g(2)$ b. $g(x) + 1$ c. $g(x^2) - g(y^2)$ d. $g(x + 1)$

   e. $g(x^2 - y^2)$

2. Given that $f(x) = x^2 - 1$, what is the domain and range of f? Find each of the following expressions:

   a. $f(3) + f(5) = 32$ b. $f(2a + 4)$ c. $f(x + 1)$ d. $2 f(a - 2)$

3. Let $f(x) = 3x^2 + 2$. Find each of the following:

   a. $f(-1)$ b. $f(0)$ c. $f(1)$ d. $f(2/3)$ e. $f(a)$ f. $f(b)$

III. Converses and Inverses: To form the converse of a relation, simply reverse the order of the components of each ordered pair. R_C is read "R Converse".

Definition: $R_C = \{(x,y): (y,x) \in R\}$

Examples: $R = \{(2,1), (3,4), (7,8)\}$ $R_C = \{(1,2), (4,3), (8,7)\}$

$R = \{(x,y): y = 3x - 2\}$ $R_C = \{(x,y): x = 3y - 2\}$

Every relation has a converse. Every function has a converse. But -- inverse is a different story! Only functions have inverses and the inverse is the converse under the condition that the converse is a function. Some functions have inverses and some don't.

Notation: f^{-1} is the notation used to identify the inverse of function f.

Definition: $f^{-1} = \{(y,x): (x,y) \in f \text{ and } \forall_y \text{ there is a unique } x\}$.

Graphically, consider all horizontal lines crossing the graph of f(x). If any line touches more than one point of f(x), then $f^{-1}(x)$ does not exist.

Assignment # 3:

1. Let $f(x) = x^3 + 1$. Find:

   a. $f^{-1}(x)$ b. $f^{-1}(5)$ c. $f^{-1}(10)$ d. $f^{-1}(0)$ e. $f^{-1}(-5)$

2. $F(x) = 2x + 1$. Find F^{-1}.

3. $F(x) = -5x + 2$. Find F^{-1}.

4. $F(x) = |x + 1|$. Does F^{-1} exist? Explain.

5. a. Use the graph of $F(x) = -3x^2 + 1$ to show that F^{-1} does not exist.

   b. Show that $F(x) = -3x^2 + 1$, $x \in (-\infty, 0]$ has an inverse function F^{-1}. Find F^{-1}.

   c. Explain the difference between the two functions in parts a and b.

Assignment # 3, Continued:

6. Examine the graphs of each of the following functions to determine whether or not F^{-1} exists. If F^{-1} exists, find it and graph it on the same coordinate system as the one containing the graph of F.

a. $F(x) = 3$ b. $F(x) = 3/x$ c. $F(x) = 1 - 3x$

d. $F = \{(x,y): y = x^2 + 2, x \geq 2\}$ e. $F(x) = 7x + 5$

f. $F = \left\{(x,y): y = \frac{1}{|x|}\right\}$ g. $F(x) = x^3 + 5$

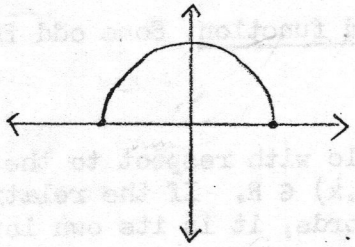
7. Examine the graphs of the given functions to determine whether or not the function has an inverse function F^{-1}. If F^{-1} exists, find it an graph F^{-1} on the same coordinate system as the one containing the graph of the given function F.

a. $F(x) = \frac{1}{2}x + 2$ b. $F(x) = 3|x|$ c. $F = \left\{(x,y): y = \frac{1}{x-1}\right\}$

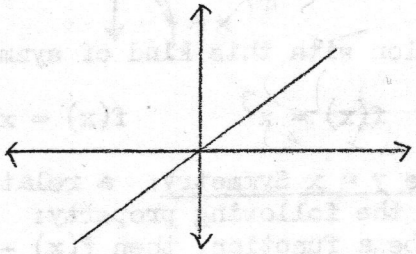
d. $F(x) = \sqrt{x}$ e. $F = \{(x,y): y = -3x^4\}$

8. Which of the following graphs represent functions and which of the functions have inverses?

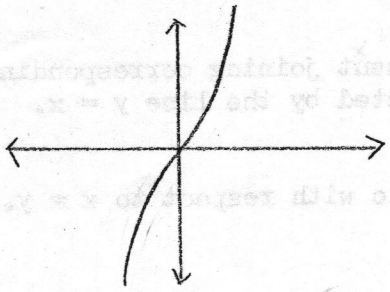
a.



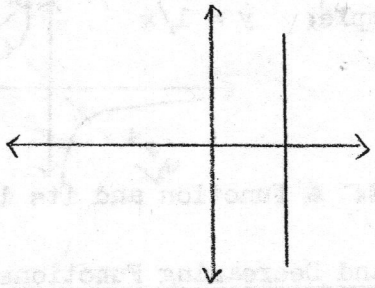
b.



c.



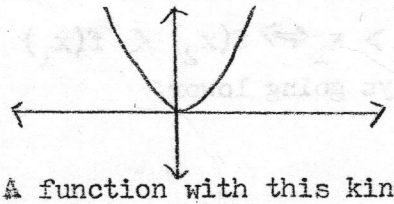
d.



IV. Symmetry: Relations can have numerous kinds of symmetry. We shall discuss 4 kinds.

1. Y-axis Symmetry: a relation which is symmetric with respect to the Y-axis has the following property: if $(x,y) \in R$ then $(-x,y) \in R$. If the relation happens to be a function, then $f(x) = f(-x)$.

Example: $y = x^2$ Any segment joining corresponding points is bisected by the Y-axis.

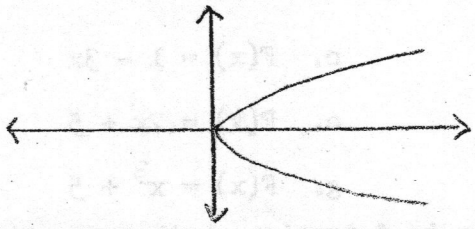


A function with this kind of symmetry is called an even function. Some even functions are:

$f(x) = x^2$ $f(x) = x^4$ $f(x) = |x|$

2. X-axis Symmetry: a relation which is symmetric with respect to the X-axis has the following porperty: if $(x,y) \in R$ then $(x, -y) \in R$. No relation with X-axis symmetry is a function, except for $f(x) = 0$.

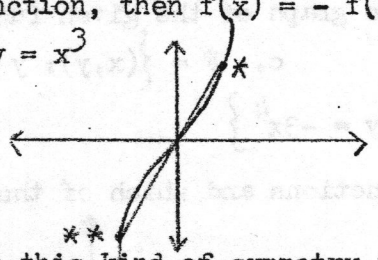
Example: $y^2 = x$



Any segment joining corresponding points is bisected by the X-axis.

3. Origin Symmetry: a relation which is symmetric with respect to the origin has the following property: if $(x,y) \in R$ then $(-x,-y) \in R$. If the relation happens to be a function, then $f(x) = -f(-x)$.

Example: $y = x^3$



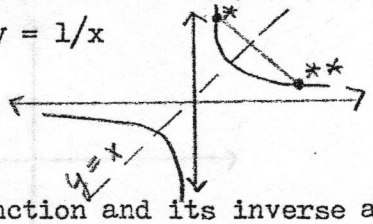
Any segment joining corresponding points is bisected by the origin. See (*,**)

A function with this kind of symmetry is called an odd function. Some odd functions are:

$f(x) = x^3$ $\qquad$ $f(x) = x^5$ $\qquad$ $f(x) = x$

4. Line y = x Symmetry: a relation which is symmetric with respect to the line $y = x$ has the following property: if $(x,y) \in R$ then $(y,x) \in R$. If the relation happens to be a function, then $f(x) = f^{-1}(x)$. In other words, it is its own inverse.

Example: $y = 1/x$



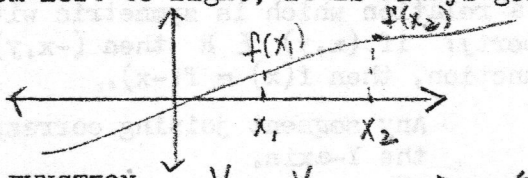
Any segment joining corresponding points is bisected by the line $y = x$. See (*,**).

NOTE: A function and its inverse are symmetric with respect to $x = y$.

Increasing and Decreasing Functions:

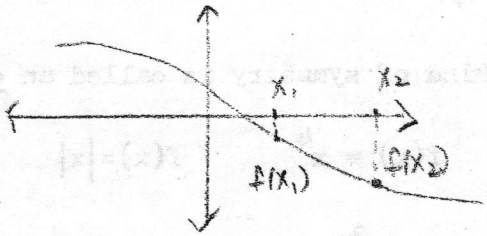
1. A STRICTLY INCREASING FUNCTION: $\forall_{x_1} \forall_{x_2} \quad x_2 > x_1 \iff f(x_2) > f(x_1)$

Reading the graph from left to right, it is always going higher.



2. A STRICTLY DECREASING FUNCTION: $\forall_{x_1} \forall_{x_2} \quad x_2 > x_1 \iff f(x_2) < f(x_1)$

Reading the graph from left to right, it is always going lower.



Assignment # 4:

1. For each of the following, graph, identify any symmetry (of the 4 types discussed), and find the domain and range:

   a. $y = |2x + 11|$ b. $y = |3x| + 3x$ c. $y^3 = 4x$

   d. $y = -x^2/2$ e. $y = |x| + x$ f. $x^2 + 4y^2 = 36$

2. Identify each of the following functions as even or odd:

   a. $y = 4$ b. $y = |x|^2$ c. $y = 2x^3$

   d. $y = 4x$ e. $y = \sqrt{x^2 - 1}$ f. $y = \frac{1}{x^2}$

3. Identify each of the following functions as strictly increasing, strictly decreasing, or neither.

   a. $y = 4x + 1$ b. $y = 3$ c. $y = x^2$

   d. $y = x^3$ e. $y = \sqrt{x}$ $y = 1/x$

4. For each of the following relations R, graph R, find R_C, and graph R amd R_C on the same coordinate system.

   a. $\{(x,y): 4x^2 + y^2 = 16\}$ b. $\{(x,y): y = x^3\}$

   c. $\{(x,y): y = 2x + 4\}$ d. $\{(3,2), (8,0), (4,1), (5,2)\}$

NOW: (1) Study the objectives for this L.A.P.
(2) Take the Trial Run.
(3) Take the L.A.P. Test

ALGEBRA 2 - FUNCTIONS

BEHAVIORAL OBJECTIVES

I. Given a relation

- A. Graph it
- B. Determine its
  1. Domain
  2. Range
- C. Discuss its symmetry
- D. Determine whether or ~~not the relation is a fu~~nction

II. Define

- A. Function
- B. Domain of a function
- C. Range of a function
- D. The inverse of a function

III. Given a function

- A. Graph it
- B. Determine its
  1. Domain
  2. Range
- C. Discuss its symmetry (if it has such: X, Y, Origin, or y = x)
- D. Identify it as
  1. Odd
  2. Even
  3. Neither
- E. Determine its inverse if it has one
- F. Identify it as increasing, decreasing, or neither

IV. Use the vertical line test to identify a function.

V. Use the horizontal line test to determine whether or not a function has an inverse.

I. Define: (a) Function (b) Domain of a function (c) Range of a function
(c) Relation which has X-axis symmetry, Y-axis symmetry, origin symmetry

II. If $R = \{(6,5), (2,8), (3,9)\}$ then $R_C =$ _____.

III. Evaluate:

(a) $f(x) = \sqrt{x+1}$; $f(8) =$ _____

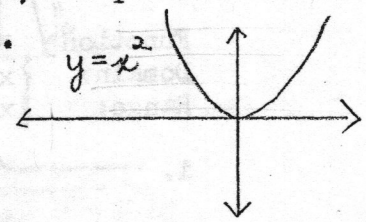
(b) $f(x) = x^2 - 2$; $f(a) =$ _____

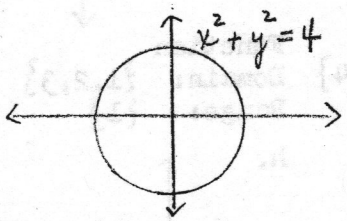
(c) $f(x) = \frac{x}{4} + 2$; $f(10) =$ _____

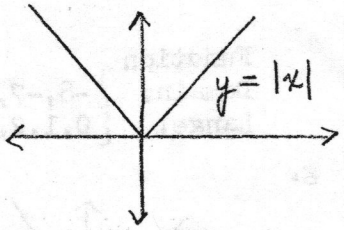
(d) $f(x) = x^2 - 3$; $f(a^2) =$ _____
$f(2) =$ _____
$f(-2) =$ _____

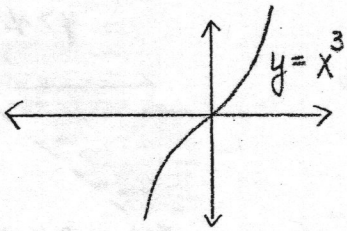
IV. Which of the following graphs represents:

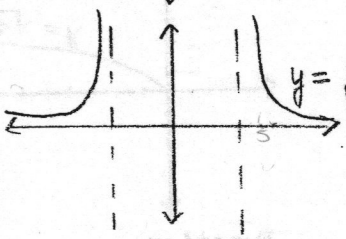
(a) a function?
(b) a function with an inverse?
(c) a relation with X-axis symmetry?
(d) a relation with Y-axis symmetry?
(e) a relation with origin symmetry?
(f) a relation which is symmetric with respect to the line y = x?
(g) an odd function?
(h) an even function?
(i) a function which is monotonic increasing over some positive domain?
(j) a function which is monotonic decreasing over some positive domain?
(k) Give the domain and range of each.
(l) Graph the converse of each.

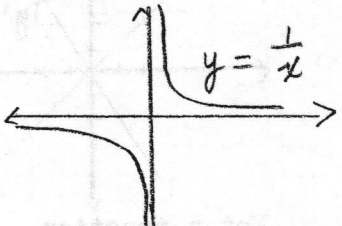
1. 

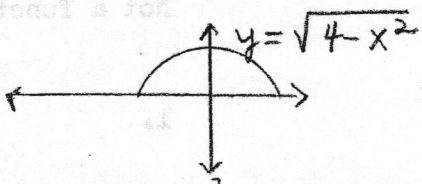
2. 

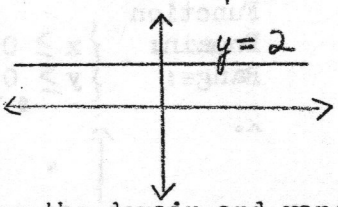
3. 

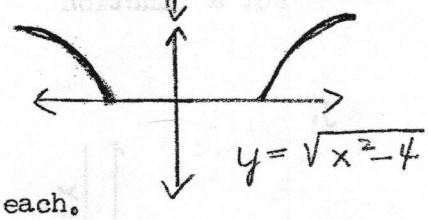
4. 

5. 

6. 

7. 

8. 

9. 

V. Graph f, graph f^{-1}, and determine the domain and range of each.

(a) $f(x) = 3x + 2$

(b) $f(x) = \sqrt{x}$

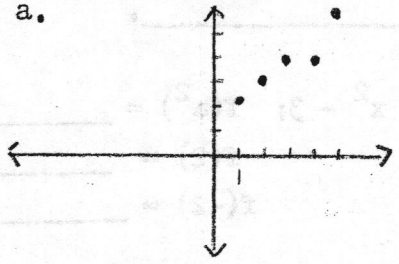
(c) $f(x) = x^3 + 1$

(d) $f(x) = -1/x$

ALGEBRA 2 - FUNCTIONS
ANSWERS

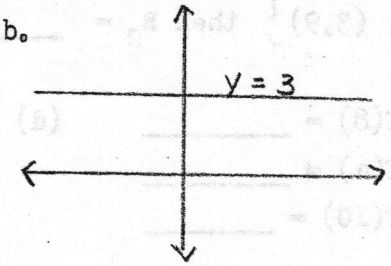
Assignment # 1

a.



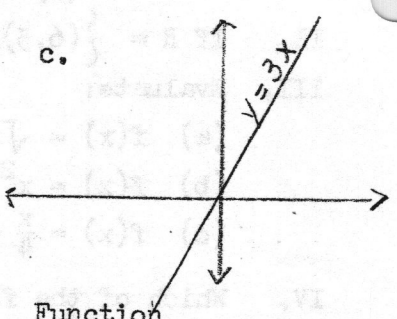
Function
Domain: $\{1,2,3,4,5\}$
Range: $\{2,3,4,6\}$

b.



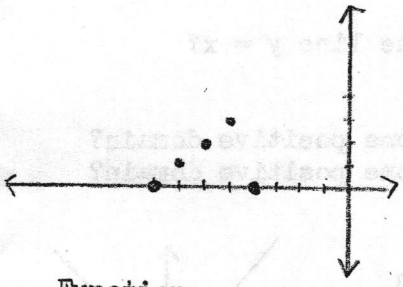
Function
Domain: $\{x \in R\}$
Range: $\{3\}$

c.



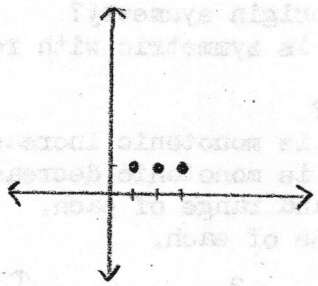
Function
Domain: $\{x \in R\}$
Range $\{x \in R\}$

d.



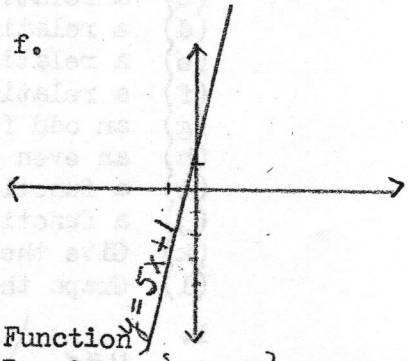
Function
Domain: $\{-8,-7,-6,-5,-4\}$
Range: $\{0,1,2,3\}$

e.



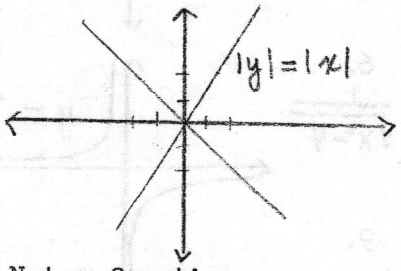
Function
Domain: $\{1,2,3\}$
Range: $\{1\}$

f.



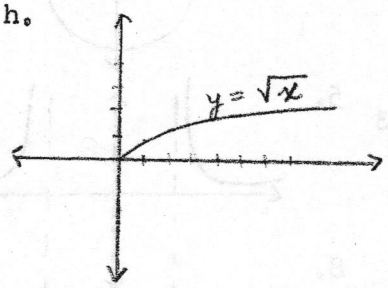
Function
Domain: $\{x \in R\}$
Range: $\{x \in R\}$

g.



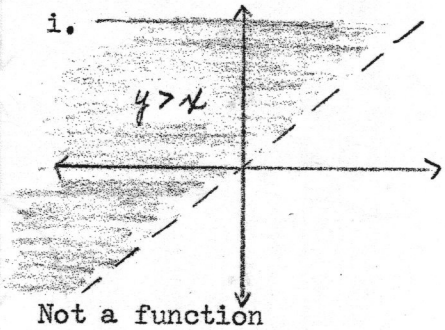
Not a function

h.



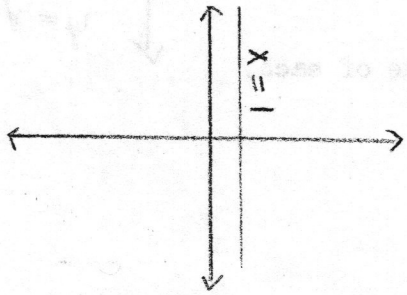
Function
Domain: $\{x \geq 0\}$
Range: $\{y \geq 0\}$

i.



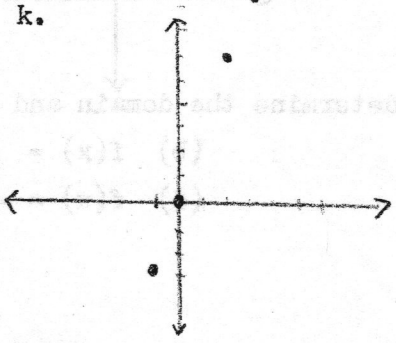
Not a function

j.



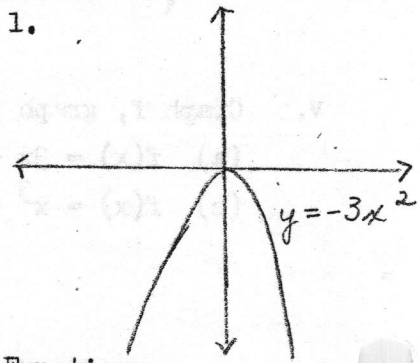
Not a function

k.



Function
Domain: $\{-1,0,2,3\}$
Range: $\{-3,0,6,9\}$

l.

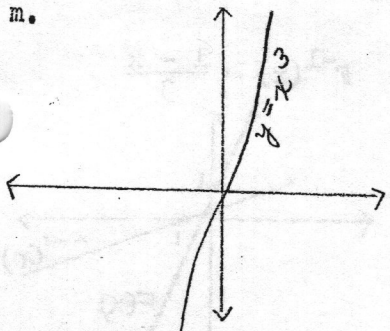


Function
Domain: $\{x \in R\}$
Range: $\{y \leq 0\}$

ANSWERS

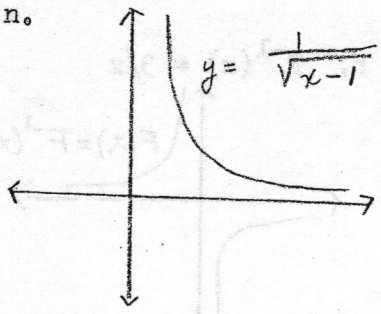
Assignment # 1 Continued:

m.



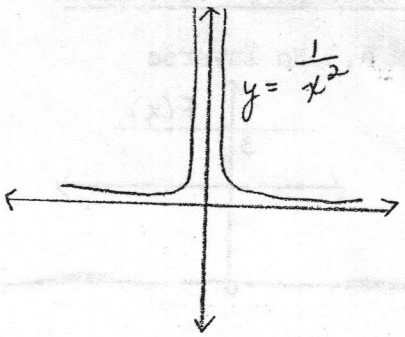
Function
Domain: $\{x \in R\}$
Range: $\{ \in R\}$

n.



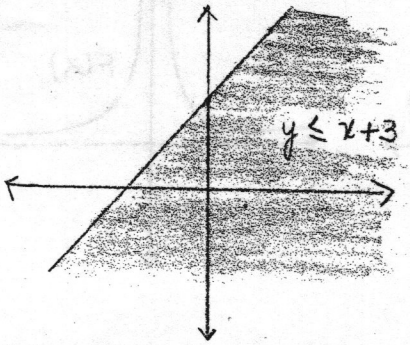
Function
Domain: $\{x > 1\}$
Range: $\{y > 0\}$

o.



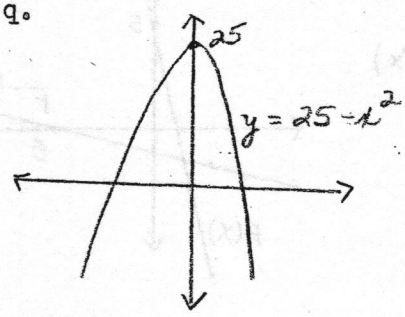
Function
Domain: $\{x \neq 0\}$
Range: $\{y > 0\}$

p.



Not a function

q.



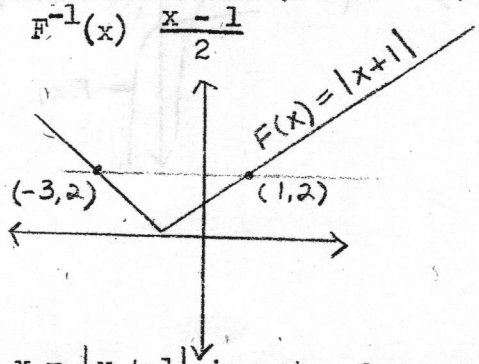
Function
Domain: $\{x \in R\}$
Range: $\{y \leq 25\}$

Assignment # 2:

1. a. 5 b. $x^2 + 2$ c. $x^4 - y^4$ d. $x^2 + 2x + 2$ e. $x^4 - 2x^2y^2 + y^4 + 1$
2. Domain of f: $\{x \in R\}$ Range of f: $\{f(x) \geq -1\}$
   a. 32 b. $4a^2 + 16a + 15$ c. $x^2 + 2x$ d. $2a^2 - 8a + 6$
3. a. 5 b. 2 c. 5 d. 10/3 e. $3a^2 + 2$ f. $3b^2 + 2$

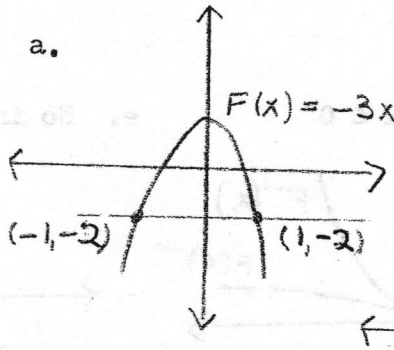
Assignment # 3:

1. a. $f^{-1}(x) = \sqrt[3]{x - 1}$ b. $\sqrt[3]{4}$ c. $\sqrt[3]{9}$ d. -1 e. $\sqrt[3]{-6}$
2. $F^{-1}(x)$ $\frac{x - 1}{2}$
3. $F^{-1}(x) = \frac{2 - x}{5}$ or $\frac{x - 2}{-5}$
4.

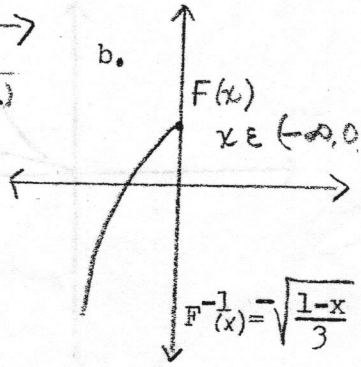


$x = |y + 1|$ is not a function.

5. a.

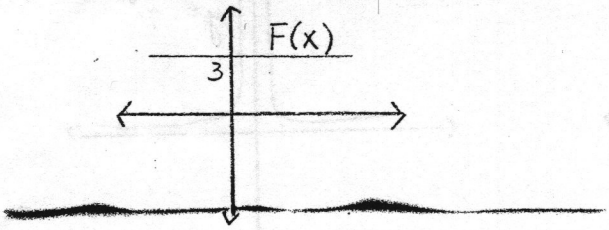


b.

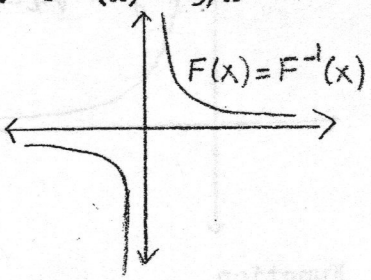


ANSWERS
ASSIGNMENT #3 Continued:

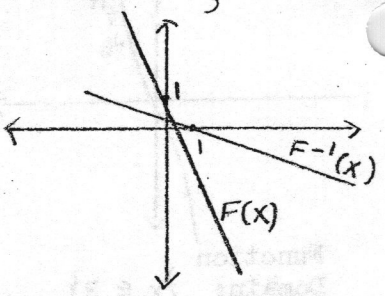
6. a. No Inverse



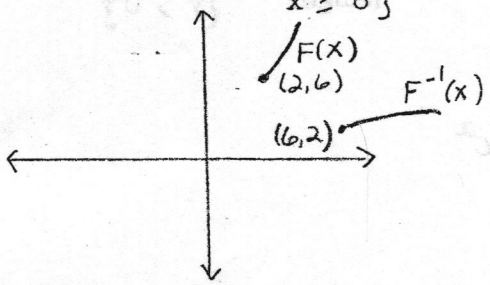
b. $F^{-1}(x) = 3/x$



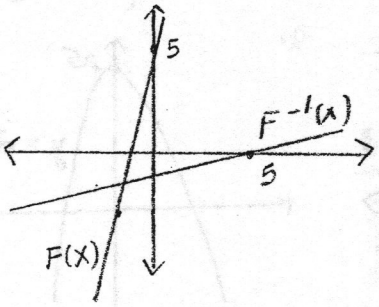
c. $F^{-1}(x) = \frac{1 - x}{3}$



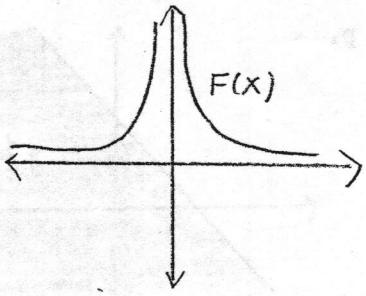
d. $F^{-1} = \{(x,y): y = \sqrt{x - 2}\ \ x \geq 6\}$



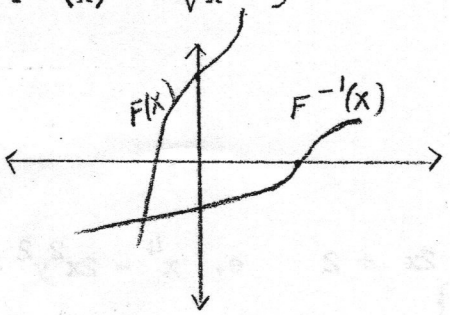
e. $F^{-1}(x) = \frac{x - 5}{7}$



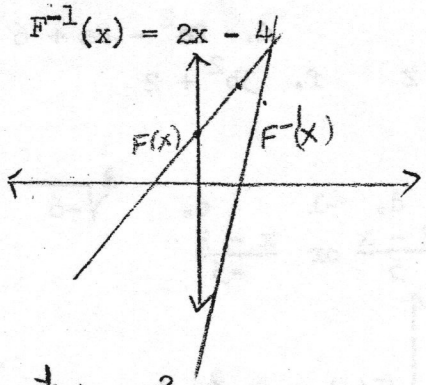
f. No inverse



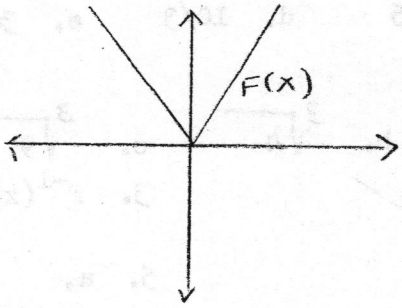
g. $F^{-1}(x) = \sqrt[3]{x - 5}$



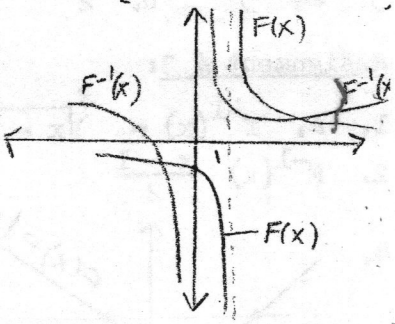
7. a. $F^{-1}(x) = 2x - 4$



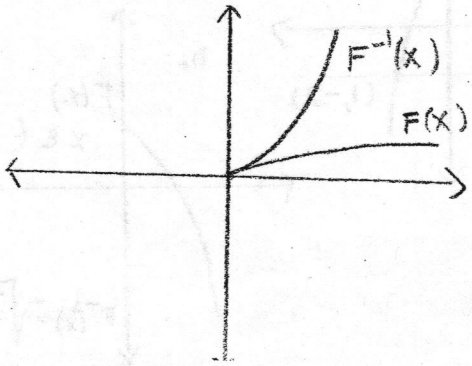
b. No inverse



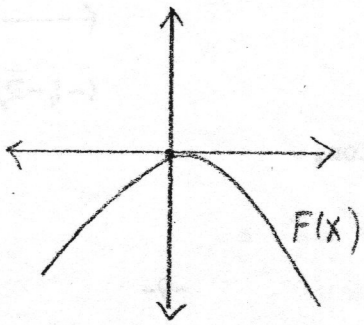
c. $F^{-1} = \{(x,y): y = \frac{1}{x} + 1$



d. $F^{-1}(x) = x^2 \quad x \geq 0$



e. No inverse

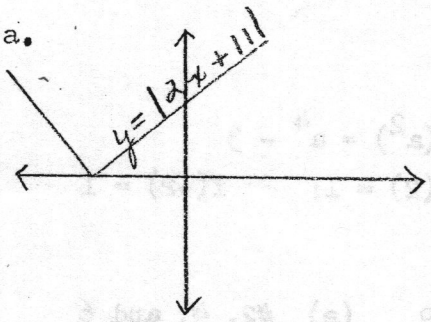


ANSWERS
ASSIGNMENT # 3 Continued:

8. a., b., and c. are functions. Only b. and c. have inverses.

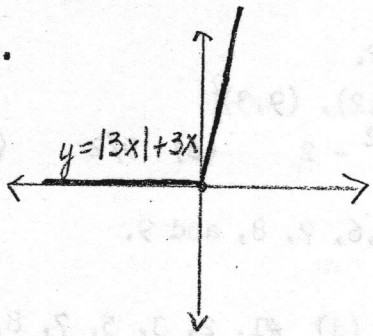
ASSIGNMENT # 4:

1. a.



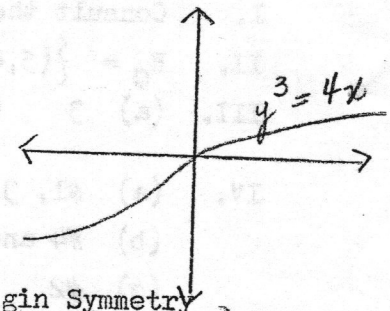
No symmetry
Domain: $\{x \in R\}$
Range: $\{y \geq 0\}$

b.



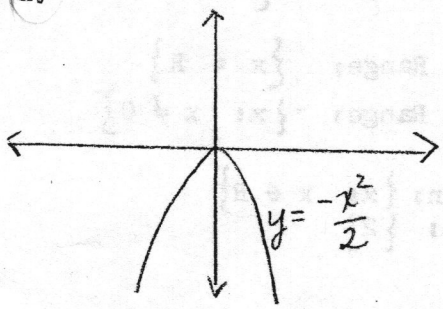
No symmetry
Domain: $\{x \in R\}$
Range: $\{y \geq 0\}$

c.



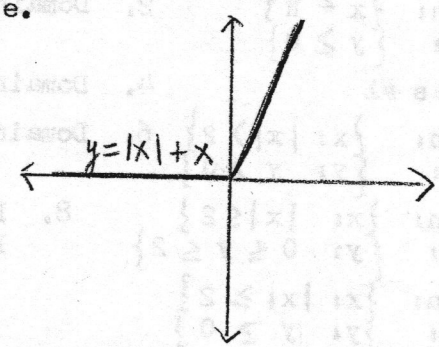
Origin Symmetry
Domain: $\{x \in R\}$
Range: $\{y \in R\}$

d.



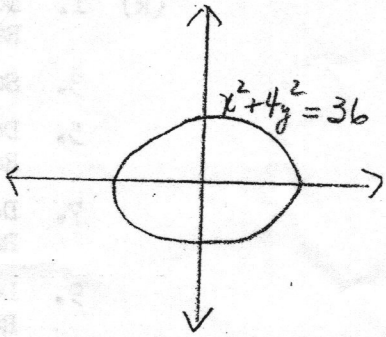
Y-axis Symmetry
Domain: $x \in R$
Range: $y \leq 0$

e.



No Symmetry
Domain: $x \in R$
Range: $y \geq 0$

f.

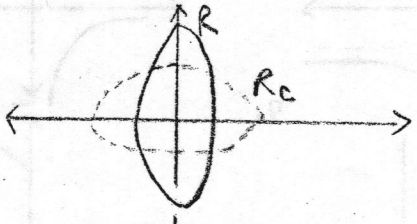


X-axis, Y-axis and Origin Symmetry
Domain: $|x| \leq 6$
Range: $|y| \leq 3$

2. a. Even b. Even c. Odd d. Odd e. Even f. Even

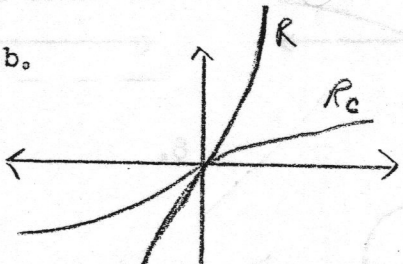
3. a. Increasing b. Neither c. Neither d. Increasing e. Increasing
f. Neither

4. a.



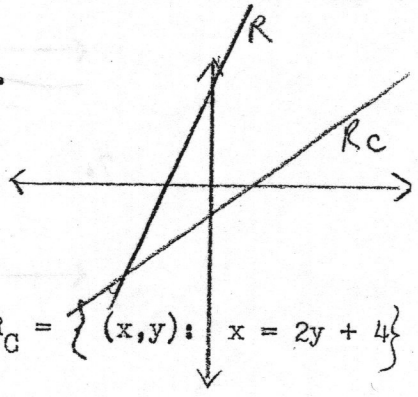
$R_C = \{(x,y): 4y^2 + x^2 = 16\}$

b.



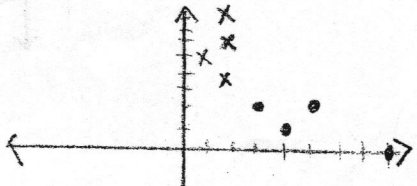
$R_C = \{(x,y): x = y^3\}$

c.



$R_C = \{(x,y): x = 2y + 4\}$

d.



$R_C = \{(2,3), (0,8), (1,4), (2,5)\}$

ANSWERS
TRIAL RUN
ALGEBRA 2 - FUNCTIONS

I. Consult the L.A.P.

II. $R_C = \{(5,6), (8,2), (9,3)\}$

III. (a) 3 (b) $a^2 - 2$ (c) 9/2 (d) $f(a^2) = a^4 - 3$
$f(2) = 1$; $f(-2) = 1$

IV. (a) #1, 3, 4, 5, 6, 7, 8, and 9.

(b) #4 and 6

(c) #2 (d) #1, 2, 3, 5, 7, 8, and 9 (e) #2, 4, and 6

(f) #2 and 6 (g) #4, and 6 (h) #1, 3, 5, 7, 8, and 9

(i) #1, 3, 4, and 9 (j) #5, 6, and 7

(k) 1. Domain: $\{x \in R\}$
Range: $\{y \geq 0\}$

2. Domain and Range: $\{x: -2 \leq x \leq 2\}$

3. Same as #1

4. Domain and Range: $\{x \in R\}$

5. Domain: $\{x: |x| > 2\}$
Range: $\{y: y > 0\}$

6. Domain and Range: $\{x: x \neq 0\}$

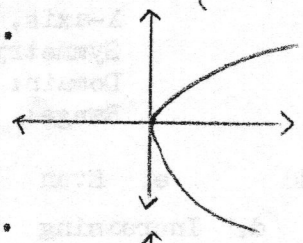
7. Domain: $\{x: |x| \leq 2\}$
Range: $\{y: 0 \leq y \leq 2\}$

8. Domain: $\{x: x \in R\}$
Range: $\{2\}$

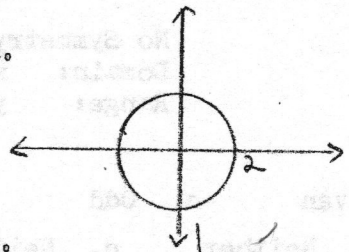
9. Domain: $\{x: |x| \geq 2\}$
Range: $\{y: y \geq 0\}$

(l)

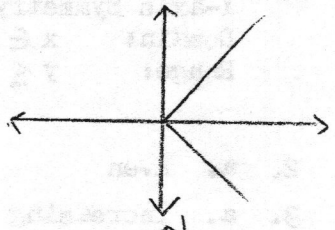
1.



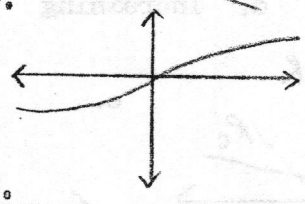
2.



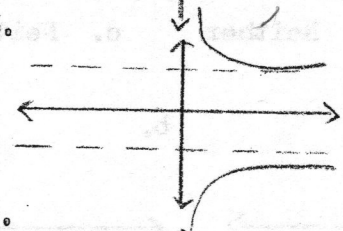
3.



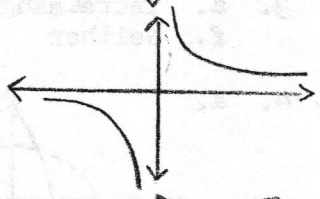
4.



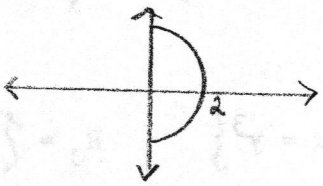
5.



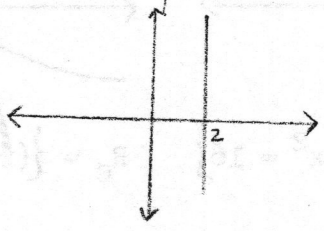
6.



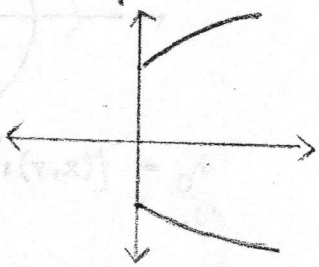
7.



8.

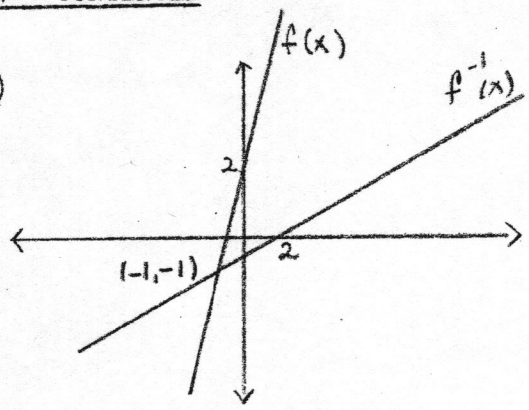


9.



ANSWERS
TRIAL RUN - CONTINUED

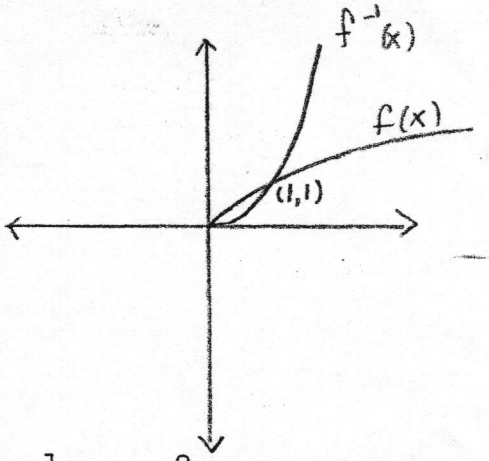
V. (a)



$f^{-1}(x) = \frac{x - 2}{3}$

Domain and Range of each: $\{x: x \in R\}$

(b)

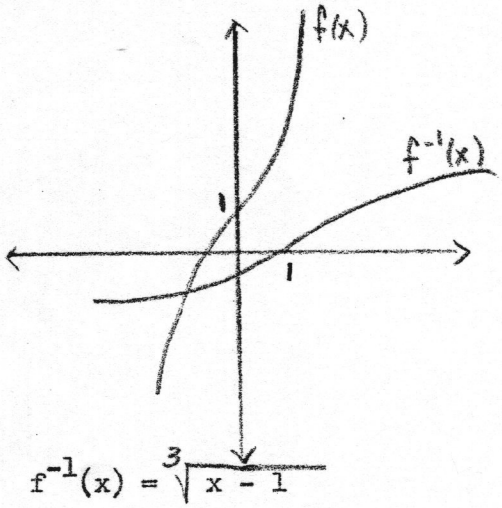


$f^{-1}(x) = x^2$ for $x \geq 0$

Domain and Range of each:

$\{x: \quad x \geq 0\}$

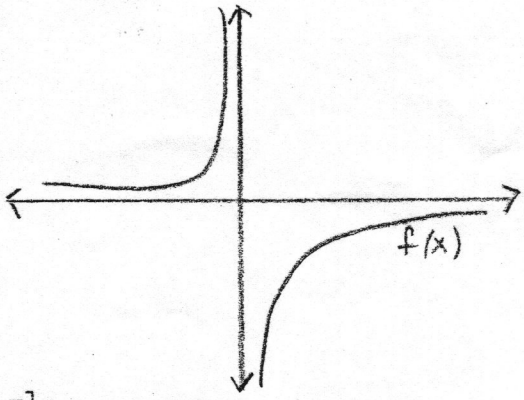
(c)



$f^{-1}(x) = \sqrt[3]{x - 1}$

Domain and Range of each: $\{x: \quad x \in R\}$

(d)



$f^{-1}(x) = f(x) = -1/x$

Domain and Range of each:

$\{x: \quad x \neq 0\}$