ALGEBRA 2

LEARNING ACTIVITIES PACKAGE

QUADRATIC EQUATIONS

OBJECTIVES

I. Define: a quadratic equation

II. Be able to write
   A. An example of a quadratic equation
   B. The quadratic formula
   C. The discriminant of the quadratic formula

III. Given a quadratic equation
   A. Write the equation in standard form
   B. Solve the equation by
      1. Factoring (when factors are obvious)
      2. Completing the square
      3. Using the quadratic formula

IV. Use the discriminant of the quadratic formula to determine whether a given quadratic equation has
   A. Two unequal, real roots
   B. Two equal, real roots
   C. Two complex roots

V. Derive the quadratic formula by using the completing the square method on the general quadratic equation

VI. Given a quadratic equation determine
   A. The sum of its roots
   B. The product of its roots

VII. Given the sum and the product of the roots of a quadratic equation, write a quadratic equation which has those roots.

SECTION 1 THE QUADRATIC EQUATION

Definition: A quadratic equation is an equation of the form: $ax^2 + bx + c = 0$, for a, b, and c real numbers, and $a \neq 0$.

EXERCISE 1 For each of the following: (a) Determine whether or not the equation is quadratic; (b) If the equation is quadratic, determine the the respective values of a, b, and c

1. $3x^2 + 4x - 5 = 0$
2. $4x^2 - 5 = 0$
3. $5x + 3x^2 = 9$
4. $(x + 1)(x - 3) = 0$
5. $x(x + 3) = 0$
6. $x^2 = 7x - 10$
7. $2x = x^2 - 3$
8. $x(x - 1)(x + 1) = 0$
9. $(x + 1)(x - 1) = x^2$
10. $(x + 3)(x - 1) = x^2$
11. $2x(x + 7) = 6$
12. $4x^2 + 10 = 5x - 1$
13. $2x(x + 3) = 2x^2$
14. $(x + 2)(x - 2) = 0$
15. $x(x + 1) = 2x - 5$

SECTION 2 SOLUTION OF QUADRATIC EQUATIONS BY FACTORING

To solve a quadratic equation by factoring:

1. Put the equation in the standard form: $ax^2 + bx + c = 0$
2. Factor the non-zero side of the equation.
3. Set each factor equal to zero. Note if $ab = 0$ then $a = 0$ or $b = 0$.
4. Solve each of the resulting equations for the variable.
5. Check the roots in the given equation.

Examples: 1. $x^2 - 2x = 15$

$x^2 - 2x - 15 = 0$

$(x - 5)(x + 3) = 0$

$x - 5 = 0$ or $x + 3 = 0$

$x = 5$ or $x = -3$

Both roots check.

The solution set is: $\{-3, 5\}$

2. $x^2 = 9$

$x^2 - 9 = 0$

$(x + 3)(x - 3) = 0$

$x + 3 = 0$ or $x - 3 = 0$

$x = -3$ or $x = 3$

Both answers check.

The solution set is: $\{-3, 3\}$

EXERCISE 2 Solve each of the following quadratic equations by factoring:

1. $x^2 - 5x + 4 = 0$
2. $x^2 + 10 = 7x$
3. $x^2 - x - 6 = 0$
4. $9x - 10 = 2x^2$
5. $2x^2 + x = 15$
6. $1 = \frac{1}{2}x^2 + 1$
7. $25x^2 = 16$
8. $9x - 10 = 2x^2$
9. $x^2 + 7x = -10$
10. $2x^2 + x = x^2 + 6$
11. $4(25 - x^2) = 9x^2$
12. $3x + 10 = x^2 + 4x + 4$
13. $x^2 = 1$
14. $x^2 - 2x = 24$
15. $\frac{9}{4}x = \frac{1}{2}x^2 + 1$
16. $2x + 4 = x^2 - x$
17. $2x^2 - 9x = 5$
18. $25x^2 = 20x - 4$

SECTION 3 SOLUTION OF QUADRATIC EQUATIONS BY COMPLETING THE SQUARE

Some quadratic equations are difficult to factor. If such is the case, the above method is not helpful. The method discussed in this section can be applied to any quadratic equation. In addition to being useful in this section of the L.A.P., the method of completing the square is used over and over in higher mathematics. The management suggests that you become an expert at completing the square.

For example: solve for x: $3x^2 - 2x - 2 = 0$

THE FACTORS OF $3x^2 - 2x - 2$ ARE JUST NOT OBVIOUS!

To solve: $3x^2 - 2x - 2 = 0$, by the method of completing the square:

1. Isolate the 'x' terms: $3x^2 - 2x = 2$
2. Divide by the coefficient of x^2: $x^2 - \frac{2}{3}x = \frac{2}{3}$
3. Add $(\frac{1}{2}$ the coefficient of the 'x' term$)^2$ to both sides: $x^2 - \frac{2}{3}x + \frac{1}{9} = \frac{2}{3} + \frac{1}{9}$
4. The left hand of the equation is a perfect square: $(x - \frac{1}{3})^2 = \frac{7}{9}$
5. Take the square root of both sides: $\left|x - \frac{1}{3}\right| = \frac{\sqrt{7}}{3}$
6. Simplify: $x - \frac{1}{3} = \pm\frac{\sqrt{7}}{3}$
7. Solve for x: $x = \frac{1}{3} \pm \frac{\sqrt{7}}{3}$
8. Write in a simple form: $x = \frac{1 \pm \sqrt{7}}{3}$

EXERCISE 3 Follow the above pattern religiously and solve each of the following quadratic equations by using the completing the square method:

1. $x^2 + 2x = 4$	2. $x^2 + 4x = 21$	3. $x^2 - 6x = 16$
4. $x^2 - 8x + 12 = 0$	5. $x^2 + 2x = 48$	6. $2x^2 - 10x = 1$
7. $3x^2 - 3x = 1$	8. $x^2 - 2x - 5 = 0$	9. $x^2 + 4x = 12$
10. $2x^2 - 3x - 2 = 0$	11. $10x^2 + x = 21$	12. $4x^2 - 11x = 3$
13. $2x^2 + x = 5$	14. $2x^2 + 4x - 3 = 0$	15. $x^2 = 4 - 6x$

THIS IS ABOVE AND BEYOND THE CALL OF DUTY: $ax^2 + bx + c = 0$

If you can solve the above general quadratic equation for x, you are well on your way to becomming a mathematician!

SECTION 4 SOLUTION OF QUADRATIC EQUATIONS BY THE QUADRATIC FORMULA

The general quadratic equation is: $ax^2 + bx + c = 0$

Below is the solution of the general quadratic equation by the method of completing the square. Follow it carefully. Ask about any step that you do not understand. Be able to do this on your own! Deriving the quadratic formula is an objective of this L.A.P.

$$ax^2 + bx + c = 0$$

$$ax^2 + bx = -c$$

$$x^2 + \frac{b}{a}x = -\frac{c}{a}$$

$$x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} = \frac{b^2}{4a^2} - \frac{c}{a}$$

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}$$

$$\left|x + \frac{b}{2a}\right| = \frac{\sqrt{b^2 - 4ac}}{|2a|}$$

$$x + \frac{b}{2a} = \pm\frac{\sqrt{b^2 - 4ac}}{2a}$$

THIS IS THE QUADRATIC FORMULA

$$\boxed{x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}}$$

The quadratic formula is a great tool for solving quadratic equations. Memorize the formula. Study the examples below to see how it can be used:

Examples:

1. Solve for x: $3x^2 - 2x = 5$

Put in standard form:

$$3x^2 - 2x - 5 = 0$$

$a = 3$; $b = -2$; $c = -5$

Hence:

$$x = \frac{2 \pm \sqrt{4 - 4(3)(-5)}}{6}$$

$$x = \frac{2 \pm \sqrt{64}}{6}$$

The solution set is: $\left\{\frac{5}{3}, -1\right\}$

2. Solve for x: $2x^2 - 6x + 3 = 0$

$$x = \frac{6 \pm \sqrt{36 - 4(2)(3)}}{4}$$

$$x = \frac{6 \pm \sqrt{36 - 24}}{4}$$

$$x = \frac{6 \pm \sqrt{12}}{4} = \frac{6 \pm 2\sqrt{3}}{4}$$

$$x = \frac{3 \pm \sqrt{3}}{2}$$

The solution set is: $\left\{\frac{3 \pm \sqrt{3}}{2}\right\}$

3. Solve for x: $x^2 + 4x + 5 = 0$

$$x = \frac{-4 \pm \sqrt{16 - 4(1)(5)}}{2}$$

$$x = \frac{-4 \pm \sqrt{-4}}{2}$$

$$x = \frac{-4 \pm 2i}{2}$$

$$x = -2 \pm 1$$

The solution set is: $\{-2 \pm i\}$

4. Solve for x: $5x^2 + 7x = -3$

$$5x^2 + 7x + 3 = 0$$

$$x = \frac{-7 \pm \sqrt{49 - 60}}{10}$$

$$x = \frac{-7 \pm \sqrt{-11}}{10}$$

The solution set is: $\left\{ \frac{-7 \pm i\sqrt{11}}{10} \right\}$

EXERCISE 4 Solve each of the following quadratic equations by using the quadratic formula:

1. $x^2 - 3x - 3 = 0$	2. $2x^2 + 7x + 1 = 0$	3. $3x^2 = 5x + 4$
4. $3 = 2x^2 + 4x$	5. $2x^2 - 5x + 1 = 0$	6. $x^2 + 2x + 4 = 0$
7. $3x^2 + 2x = -6$	8. $2x^2 - 10x = 9$	9. $2x^2 - x + 8 = 0$
10. $2x^2 - 2x + 3 = 0$	11. $x^2 - 20x - 10 = 0$	12. $x^2 + 9x + 12 = 0$
13. $x^2 + 4x + 16 = 0$	14. $x^2 = -6x + 4$	15. $3x(x - 1) = 1$

SECTION 5 THE DISCRIMINANT OF THE QUADRATIC EQUATION

From the quadratic formula, the expression: $b^2 - 4ac$ is called the discriminant. The value of the discriminant will determine whether the roots of the given equation are real numbers or imaginary numbers.

Case 1: $b^2 - 4ac > 0$	The roots are distinct and real numbers. Note, in the numerator of the quadratic formula, once the $\sqrt{b^2 + 4ac}$ is added and once subtracted.
Case 2: $b^2 - 4ac = 0$	The roots are real and equal. Note, the quadratic formula simplifies to $-b/2a$
Case 3: $b^2 - 4ac < 0$	The roots are complex numbers. WHY?

Every quadratic equation has two roots. The roots are: (a) two non-equal, real; (b) two equal, real; or (c) two complex.

EXERCISE 5

A. Determine the nature of the roots of a quadratic equation, given the value of the discriminant.
(a) real and unequal; (b) real and equal (c) complex

1. 49	2. 0	3. 16	4. -4	5. 17	6. 100
7. -13	8. 86	9. 54	10. -121	11. 33	12. -25

B. Determine the nature of the roots for each of the following. Do not solve the equations.

1. $x^2 - 7x + 10 = 0$	2. $2x^2 - 5x = 4$	3. $x^2 + 3x + 4 = 0$
4. $3x^2 - 7x - 4 = 0$	5. $9x^2 = 6x$	6. $25x^2 + 4 = 0$
7. $2x - 1 = x^2$	8. $5x^2 + 2 = 11x$	9. $4x^2 = 3x - 5$

SECTION 6 THE SUM AND THE PRODUCT OF THE ROOTS OF A QUADRATIC EQUATION

The purpose of this section is to form a quadratic equation which has specific given roots. For example: Suppose the roots of a quadratic equation are 3 and -1. Find a quadratic equation which has these roots.

Solution: If 3 and -1 are roots then $x = 3$ or $x = -1$

If $x = 3$ then $x - 3 = 0$

If $x = -1$ then $x + 1 = 0$

So: $(x - 3)(x + 1) = 0$. Hence an equation is: $x^2 - 2x - 3 = 0$

In general: Given: the roots of a quadratic equation are r_1 and r_2.

Find: A quadratic equation which has roots r_1 and r_2.

Solution: If r_1 and r_2 are roots, then $x = r_1$ or $x = r_2$

If $x = r_1$, then $x - r_1 = 0$

If $x = r_2$, then $x - r_2 = 0$

So: $(x - r_1)(x - r_2) = 0$. Hence an equation with roots r_1 and r_2 is:

$$x^2 - (r_1 + r_2)x + r_1 r_2 = 0$$

Note: The coefficient of the x term is the opposite of the sum of the roots.
The constant term is the product of the roots.

EXERCISE 6

A. Write a quadratic equation which has roots:

1. 5,2	2. 3,7	3. -1,-3	4. -2,7
5. $\frac{2}{3}$,3	6. $2 \pm \sqrt{5}$	7. $3 \pm \sqrt{7}$	8. $\pm 2i$
9. $4 \pm i\sqrt{2}$	10. $5 \pm i\sqrt{7}$	11. 0,5	12. $5 \pm i$

B. Write a quadratic equation. The sum and product of the roots are respectively:

1. 9 and 4	2. 7 and 1	3. -5 and -2	4. 1 and -3
5. -6 and 8	6. $-\frac{5}{6}$ and $\frac{1}{6}$	7. $-\frac{3}{4}$ and $\frac{3}{8}$	8. $-\frac{2}{5}$ and $\frac{1}{25}$

C. Find the sum and product of the roots of the following quadratic equations:

1. $x^2 - 5x + 2 = 0$	2. $x^2 + 4x + 9 = 0$	3. $x^2 - 7x + 8 = 0$
4. $x^2 = 3x - 7$	5. $2x^2 = 4x - 1$	6. $3x^2 + x = -2$
7. $5x^2 = x$	8. $7x^2 - 2x = 10$	9. $-2x^2 + x = 5$

SECTION 7 EVALUATION

1. Review the objectives of this L.A.P. (Make sure that you can derive the quadratic formula.)

2. Take the Trial Run

3. Take the test.

ANSWERS

EXERCISE 1

1. Yes; 3,4,-5	2. Yes; 4,0,-5	3. Yes; 3,5,-9	4. Yes; 1,-2,-3
5. Yes; 1,3,0	6. Yes; 1,-7,10	7. Yes; 1,-2,-3	8. No
9. No	10. No	11. Yes; 2,14,-6	12. Yes; 4,-5,11
13. No	14. Yes; 1,0,-4	15. Yes; 1,-1,5	

EXERCISE 2

1. $\{1,4\}$ 2. $\{2,5\}$ 3. $\{-2,3\}$ 4. $\{2,\frac{5}{2}\}$ 5. $\{-3,\frac{5}{2}\}$ 6. $\{0\}$

7. $\{\pm\frac{4}{5}\}$ 8. $\{\frac{5}{2},2\}$ 9. $\{-5,-2\}$ 10. $\{-3,2\}$ 11. $\{\pm\frac{10\sqrt{13}}{13}\}$

12. $\{-3,2\}$ 13. $\{\pm 1\}$ 14. $\{-4,6\}$ 15. $\{\frac{1}{2},4\}$ 16. $\{-1,4\}$

17. $\{-\frac{1}{2},5\}$ 18. $\{\frac{2}{5}\}$

EXERCISE 3

1. $\{-1\pm\sqrt{5}\}$ 2. $\{3,-7\}$ 3. $\{-2,8\}$ 4. $\{2,6\}$ 5. $\{-8,6\}$

6. $\{\frac{5\pm 3\sqrt{3}}{2}\}$ 7. $\{\frac{3\pm\sqrt{21}}{6}\}$ 8. $\{1\pm\sqrt{6}\}$ 9. $\{-6,2\}$ 10. $\{-\frac{1}{2},2\}$

11. $\{\frac{-3}{2}\ \frac{7}{5}\}$ 12. $\{\frac{-1}{4},3\}$ 13. $\{\frac{-1\pm\sqrt{41}}{4}\}$ 14. $\{\frac{-2\pm\sqrt{10}}{2}\}$ 15. $\{-3\pm\sqrt{13}\}$

EXERCISE 4

1. $\{\frac{3\pm\sqrt{21}}{2}\}$ 2. $\{\frac{-7\pm\sqrt{41}}{4}\}$ 3. $\{\frac{5\pm\sqrt{73}}{6}\}$ 4. $\{\frac{-2\pm\sqrt{10}}{2}\}$ 5. $\{\frac{5\pm\sqrt{17}}{4}\}$

6. $\{-1\pm i\sqrt{3}\}$ 7. $\{\frac{-1\pm i\sqrt{17}}{3}\}$ 8. $\{\frac{5\pm\sqrt{43}}{2}\}$ 9. $\{\frac{1\pm 3i\sqrt{7}}{4}\}$ 10. $\{\frac{1\pm i\sqrt{5}}{2}\}$

11. $\{10\pm\sqrt{110}\}$ 12. $\{\frac{-9\pm\sqrt{33}}{2}\}$ 13. $\{-2\pm 2i\sqrt{3}\}$ 14. $\{-3\pm\sqrt{13}\}$ 15. $\{\frac{3\pm\sqrt{21}}{6}\}$

EXERCISE 5

A.

1. a 2. b 3. a 4. c 5. a 6. a 7. c 8. a 9. a 10. c 11. a 12. c

B. 1. a 2. a 3. c 4. a 5. a 6. c 7. b 8. a 9. c

EXERCISE 6

A. 1. $x^2 - 7x + 10 = 0$ 2. $x^2 - 10x + 21 = 0$ 3. $x^2 + 4x + 3 = 0$

4. $x^2 - 5x - 14 = 0$ 5. $3x^2 - 11x + 6 = 0$ 6. $x^2 - 4x - 1 = 0$

7. $x^2 - 6x + 2 = 0$ 8. $x^2 + 4 = 0$ 9 $x^2 - 8x + 16 = 0$

10. $x^2 + 10x + 32 = 0$ 11. $x^2 - 5x = 0$ 12. $x^2 - 10x + 26 = 0$

B. 1. $x^2 - 9x + 4 = 0$ 2. $x^2 - 7x + 1 = 0$ 3. $x^2 + 5x - 2 = 0$

4. $x^2 - x - 3 = 0$ 5. $x^2 + 6x + 8 = 0$ 6. $6x^2 + 5x + 1 = 0$

7. $8x^2 + 6x + 3 = 0$ 8. $25x^2 + 10x + 1 = 0$

C. 1. $\{5,2\}$ 2. $\{-4,9\}$

3. $\{7,8\}$ 4. $\{3,7\}$ 5. $\{2,3\}$ 6. $\{-\frac{1}{3},-\frac{1}{3}\}$ 7. $\{\frac{1}{5},1\}$ 8. $\{\frac{3}{7},-\frac{16}{7}\}$ 9. $\{\frac{1}{2},\frac{5}{2}\}$

ALGEBRA 2 TRIAL RUN

Quadratic Equations

1. Write the general quadratic equation.
2. Write the quadratic formula.
3. Write the discriminant of the quadratic formula
4. Describe the roots of a quadratic formula for
   a. Discriminant > 0
   b. Discriminant $= 0$
   c. Discriminant < 0

5. Solve each of the following quadratic equations (Show your work)
   a. Solve by factoring
      $3x^2+2x-1=0$
   b. Solve by completing the square
      $x^2+4x-2=0$
   c. Solve by completing the square
      $x^2-3x=7$
   d. Solve by completing the square
      $2x^2=8x-5$
   e. Solve by completing the square
      $3x^2+2x+1=0$
   f. Solve by using the quadratic formula
      $3x^2+x-2=0$
   G. Solve by using the quadratic formula
      $x^2=4x+7$

6. Describe the roots of the following quadratic equations (Real, Equal, Comp)
   a. $3x^2+x+1=0$
   b. $x^2-7x=2$
   c. $x^2+4x+4=0$

7. For the following determine:

	sum of the roots	product of the roots
a. $x^2-2x+1=0$		
b. $x^2=7x+8$		
c. $3x^2-2x+1=0$		

8. Write a quadratic equation which has roots
   a) 3 and 4
   b) -2 and 1
   c) $\sqrt{5}$ and $-\sqrt{5}$

9. From the general quadratic equation use the method of completing the square and derrive the quadratic formula.

TRIAL RUN
ANSWERS

1. $ax^2 + bx + c = 0$

2. $\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

3. $b^2 - 4ac$

4. a. 2 real
   b. 1 equal
   c. 2 complex

5. a. $\{1/3, -1\}$

   b. $x = -2 \pm \sqrt{6}$

   c. $\frac{3 \pm \sqrt{37}}{2}$

   d. $x = \frac{4 \pm \sqrt{6}}{2}$

   e. $\frac{-1 + i\sqrt{2}}{3}$

   f. $\{-1, 2/3\}$

   g. $2 \pm \sqrt{11}$

6. a. 2 complex
   b. 2 real
   c. 1 equal

7.

	sum of the roots	product of the roots
a.	2	1
b.	7	-8
c.	$2/3$	$1/3$

8. a. $x^2 - 7x + 12 = 0$
   b. $x^2 + x - 2 = 0$
   c. $x^2 - 5 = 0$