ALGEBRA 2

SYSTEMS OF NON-LINEAR EQUATIONS

Behavioral Objectives:

I. Given a non-linear equation:
   A. By inspection, identify its graph
   B. Graph the equation.

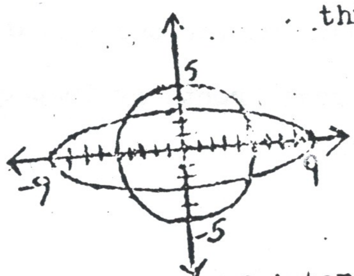
II. Given a system of two non-linear equations (or one linear and one non-linear equation):
   A. Determine the maximum number of real roots by inspection.
   B. Solve the system algebraically.

SYSTEMS OF NON-LINEAR EQUATIONS

I. Non-linear equations are those equations whose graphs are figures other than straight lines. Because they graph as circles, ellipses, parabolas, etc., a system of such equations may have more than one intersection therefore there may be multiple common roots.

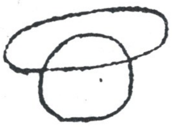
Ex. $x^2 + y^2 = 25$
$x^2 + 9y^2 = 81$

The graphs of this system look like this.



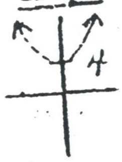
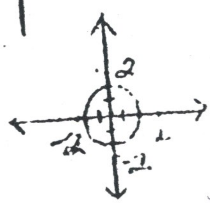
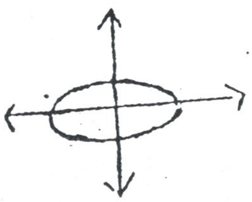
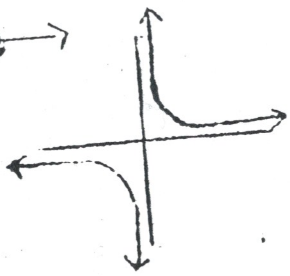
Since there are four points of intersection, the system may have up to four common roots.

(NOTE: It is possible that some of the four roots will be complex. Complex roots do not determine points of intersection. In this L.A.P., we shall focus only on intersection points. Therefore, sometimes we will find less than four roots when we solve a system containing an ellipse and a circle. An example would be a system that graphed like this:



In this system there are only two points of intersection. When we solved this system, we would find only two real roots.)

With this in mind, it will be of help to us if we learn to recognize what kind of graph an equation will make, even if we are not going to graph them. The basic kinds of non-linear graphs are:

Equation	How to recognize	Graph	Name
A) $y = x^2+4$	one variable of second degree, the other of first degree		parabola
B) $x^2+y^2 = 4$	sum of two variables, both of second degree		circle
C) $2x^2+4y^2 = 8$	same as circle, but with different coefficients		ellipse
D) $xy = 1$ (transforms to $x = \frac{1}{y}$)	two variables of degree one, multiplied		hyperbola

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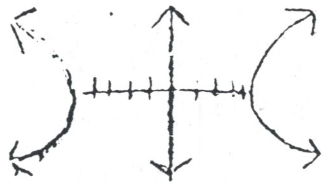
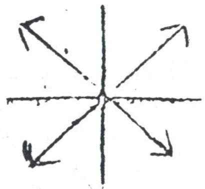
E) $x^2 - y^2 = 25$ difference of two squares



hyperbola

F) $x^2 = y^2$ two squares - equal



intersecting lines (degenerate hyperbola)

Now let's practice recognizing the kind of graph an equation will give. Do not be surprised if you meet a line graph along the way.

EXERCISE I

Look at the equations in each of these systems:

1) State what kind of graph will result from each equation.
2) Give the maximum number of intersection points of the graphs.
3) Do not graph.

EXAMPLE:

(a) $2x + 3y = 7$
(b) $5x^2 + 4y^2 = 12$

ANSWER
(a) line
(b) ellipse
2 possible intersections

1) (a) $y = x^2 + 3$
(b) $x^2 + y^2 = 4$

2) (a) $y = 2x + 5$
(b) $2x^2 + 5y^2 = 20$

3) (a) $xy = 5$
(b) $3x + 2y = 8$

4) (a) $2x = 3y^2 + 5$
(b) $3y = 5x^2 + 2$

5) (a) $x^2 + y^2 = 5$
(b) $2x^2 + 5y^2 = 250$

6) (a) $2x^2 - 5y^2 = 45$
(b) $x^2 + y^2 = 2$

7) (a) $3x^2 + 5y^2 = 75$
(b) $12x^2 + 2y^2 = 48$

8) (a) $3x = 5y^2 - 4$
(b) $2x = 3y + 4$

9) (a) $x^2 - y^2 = 0$
(b) $x = 2y + 1$

II. Now let's do more than just ~~"recognize"~~ the graphs. ~~Let's do the~~ actual graphing. Most of these can be graphed very easily now that you have learned to recognize the shapes. If we just find the x and Y intercepts and recognize the shape, that's all we need.

REMEMBER:

The value of y on the x - axis is zero The value of x on the y - axis is zero

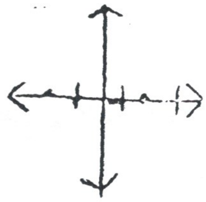
Therefore to find the x - intercepts of this equation:

$$x^2 + y^2 = 4$$

we substitute the value of 0 for y

$$x^2 + 0 = 4$$

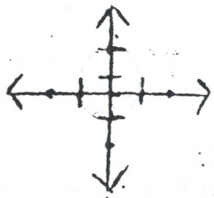
$\therefore x = \pm 2$ Plot these intercepts



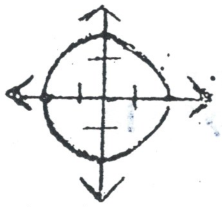
To find the y - intercepts, substitute 0 for x

$$0 + y^2 = 4$$

$\therefore y = \pm 2$ Plot these →



Since you can now recognize that $x^2+y^2 = 4$ will graph as a CIRCLE, and you already have four points of the circle, it is a simple matter to find the graph!



We graph an ELLIPSE the same way:

$$4x^2 + 9y^2 = 36$$

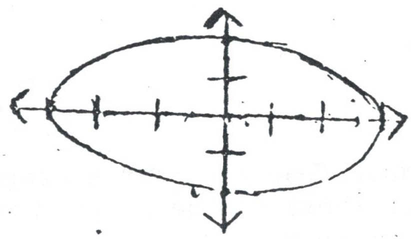
To find the x - intercepts:

$$4x^2 + 9(0) = 36$$
$$4x^2 = 36$$
$$x^2 = 9$$
$$x = \pm 3$$

To find the y - intercepts:

$$4(0) + 9y^2 = 36$$
$$9y^2 = 36$$
$$y^2 = 4$$
$$y = \pm 2$$

Graph these four points and then draw your ellipse:



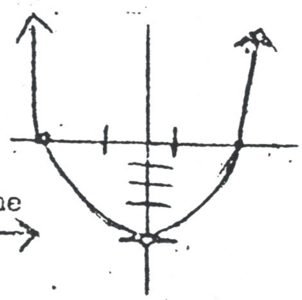
We can sometimes use the intercepts alone to graph a PARABOLA:

if $x = 0$:
$$y = x^2 - 4$$
$$y = 0 - 4$$
$$y = -4$$

if $y = 0$:
$$0 = x^2 - 4$$
$$4 = x^2$$
$$\pm 2 = x$$

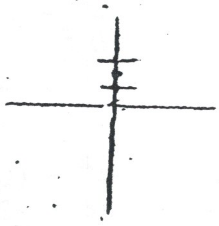
graph these:

You can finish the graph easily →



Sometimes we need more points for a PARABOLA:

y - intercept:
$$y = x^2 + 2$$
$$y = 0 + 2$$
$$y = 2$$

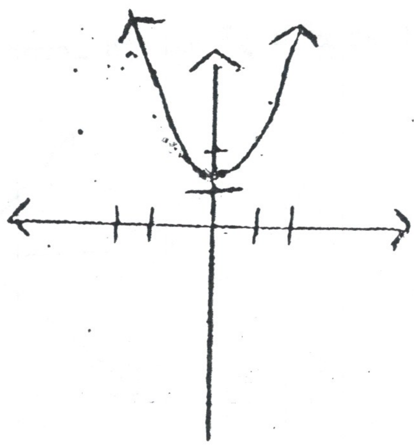


x - intercept:
$$0 = x^2 + 2$$
$$-2 = x^2$$
$$\emptyset = x$$

∴ no x-intercepts. So we try some other points:

x	y
± 1	3
± 2	6

∴ graph looks like this →



SYSTEMS OF NON-LINEAR EQUATIONS

In the graph $xy = 4$, we know we ~~have~~ a HYPERBOLA, and we know also that the graph is limited by two facts:

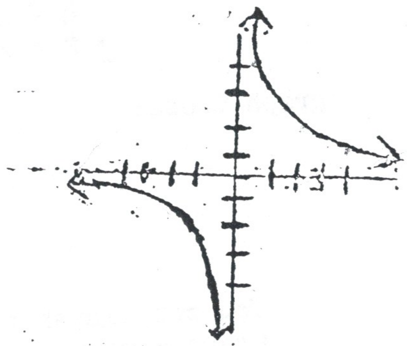
(1) neither x nor y can ever equal zero

(2) either both variables must represent positive numbers, or both must be negative (in order to get a positive product)

Therefore the graph looks like this:

You may want to plot more points:

x	y
2	2
-2	-2
1	4
4	1
-1	-4
-4	-1



Another type of HYPERBOLA is formed by the difference of two squares. In the case $x^2 - y^2 = 4$ we can find the intercepts:

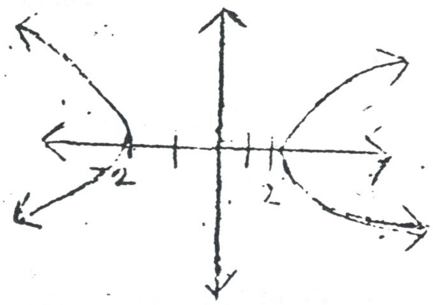
y-intercepts:

$$x^2 - y^2 = 4$$
$$(0)^2 - y^2 = -4$$
$$y^2 = -4$$
$$y = \emptyset$$

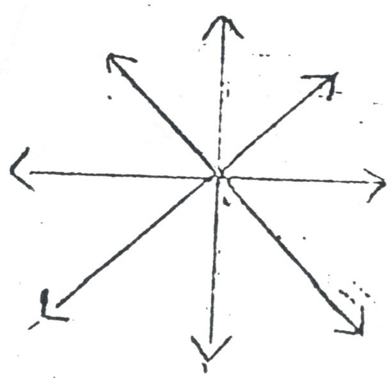
x-intercepts:

$$x^2 - (0)^2 = 4$$
$$x = \pm 2$$

Therefore the hyperbola graphs like this



Remember that $x^2 = y^2$ always forms a DEGENERATE HYPERBOLA:



SYSTEMS OF NON-LINEAR EQUATIONS

EXERCISE II.

Graph the following equations:

1) $x^2 + y^2 = 9$

2) $9x^2 + 25y^2 = 225$

3) $y = x^2 - 9$

4) $xy = 6$

5) $x^2 = y^2$

6) $y^2 - x^2 = 9$

7) $x^2 + y^2 = 1$

8) $x = y^2 - 4$

9) $4x^2 + y^2 = 4$

10) $y = x^2$ (take other points)

III. Solving systems of non-linear equations:

Since we know that solving systems of equations by graphing can only be an approximation, we have to learn how to solve these systems algebraically. To do this we apply the same methods we learned in Systems of Linear equations.

EXAMPLE: (a) $x^2 - y^2 = 9$ hyperbola
(b) $x^2 + 9y^2 = 169$ ellipse
$\therefore$ 4 possible roots.

SUBSTITUTION METHOD:

Transform (a) to solve for x^2
$x^2 = 9 + y^2$

Substitute $9+y^2$ for x^2 in (b)

$$x^2+9y^2 = 169$$
$$9+y^2+9y^2 = 169$$
$$9 + 10y^2 = 169$$
$$10y^2 = 160$$
$$y^2 = 16$$
$$y = \pm 4$$

Substitute these values for y in (a)

ADD-SUB. METHOD

(a) $x^2-y^2 = 9$
(b) $x^2+9y^2 = 169$

Subtract

$$-10y^2 = -160$$
$$y^2 = 16$$
$$y = \pm 4$$

Substitute for y in (a)

$$x^2 - 16 = 9$$
$$x^2 = 25$$
$$x = \pm 5$$

Solution Set

$$\{(5,4),(-5,4)(5,-4),(-5,-4)\}$$

If $y=4$: $x^2 - y^2 = 9$

$x^2 - (4)^2 = 9$

$x^2 - 16 = 9$

$x^2 = 25$

$x = \pm 5$

∴ 2 roots: $(5,4)$, $(-5,4)$

If $y = -4$: $x^2 - (-4)^2 = 9$

$x^2 - 16 = 9$

$x^2 = 25$

$x = \pm 5$

∴ 2 more roots: $(5,-4)$ $(-5,-4)$

Solution set: $\{(5,4), (5,-4), (-5,4), (-5,-4)\}$

<u>Be sure to check your roots in both original equations.</u>

EXAMPLE: (a) $xy = 6$ hyperbola

(b) $x^2+y^2 = 13$ circle

∴ 4 possible roots

Solve (a) for x

$xy = 6$

$x = \frac{6}{y}$

Substitute $\frac{6}{y}$ for x in (b)

$x^2+y^2 = 13$

$\left(\frac{6}{y}\right)^2+y^2 = 13$

$\frac{36}{y^2} + y^2 = 13$

Multiply by y^2: $36 + y^4 = 13y^2$

Put in standard form: $y^4-13y^2 + 36 = 0$

$(y^2-9)(y^2-4) = 0$

$y^2-9=0$ $y^2-4=0$

$y^2 = 9$ $y^2 = 4$

$y = \pm 3$ $y = \pm 2$

Substitute these values in (a)

$xy = 6$

$x = \frac{6}{y}$

$x = \frac{6}{3} = 2$

$x = \frac{6}{-3} = -2$

$x = \frac{6}{2} = 3$

$x = \frac{6}{-2} = -3$

Solution set: $\{(2,3)(-2,-3)(3,2)(-3,-2)\}$

Be sure to check your roots in both original equations.

Now it's time for you to try the "big stuff": Be sure to determine the maximum number of roots <u>before</u> you do the algebra.

EXERCISE III: Solve these systems algebraically:

1) $x^2 + y^2 = 25$
$x - y = 1$

2) $y = x^2 - 6$
$3x - y = 2$

3) $x - 2y = 0$
$x^2 - 3y^2 = 9$

4) $x^2 + y^2 = 25$
$x - y + 5 = 0$

5) $x^2 + y^2 = 5$
$y = x^2 - 3$

6) $y = \frac{1}{2}x^2$
$y = x$

7) $x^2 + y^2 = 13$
$x + y = 5$

8) $x^2 + 4y^2 = 17$
$3x^2 - y^2 = -1$

9) $x^2 + y^2 = 17$
$xy = 4$

10) $x^2 - y^2 = 7$
$2x^2 + 3y^2 = 24$

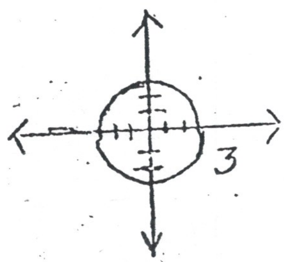
ANSWERS

EXERCISE I

1) (a) parabola (b) circle (c) 4
2) (a) line (b) ellipse (c) 2
3) (a) hyperbola (b) line (c) 2
4) (a) parabola (b) parabola (c) 4
5) (a) circle (b) ellipse (c) 4
6) (a) hyperbola (b) circle (c) 4
7) (a) ellipse (b) ellipse (c) 4
8) (a) parabola (b) line (c) 2
9) (a) intersecting lines (b) line (c) 2

EXERCISE II

1.)



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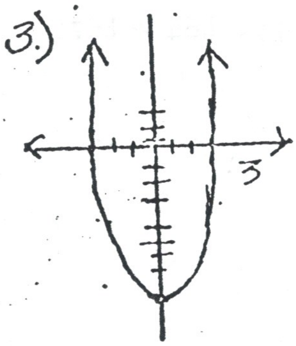
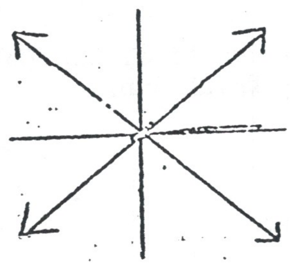


5)



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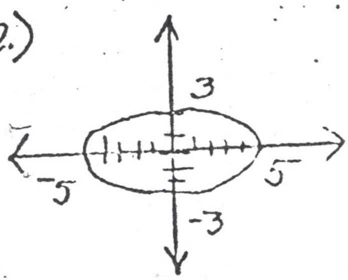

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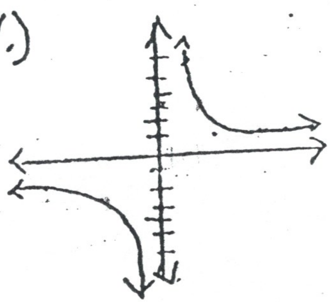

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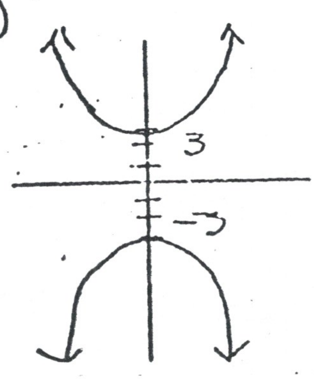

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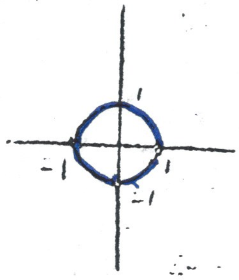

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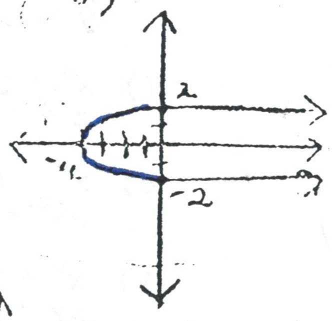

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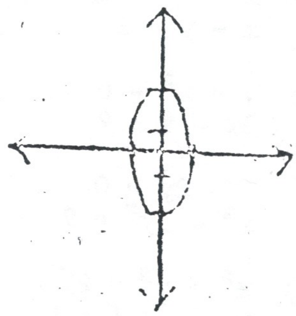

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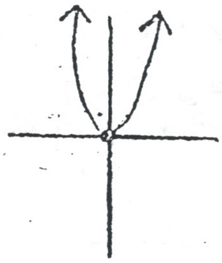


EXERCISE III

1) {(4,3) (-3,-4)}
2) {(-1,-5) (4,10)}
3) {(6,3) (-6,-3)}
4) {(-5,0) (0,5)}
5) {(2,1) (-2,1) (1,-2) (-1,-2)}
6) {(0,0) (2,2)}
7) {(2,3) (3,2)}
8) {(1,2) (1,-2) (-1,2) (-1,-2)}
9) {(1,4) (4,1) (-1,-4) (-4,-1)}
10) {$(-3,\sqrt{2})$ $(-3,-\sqrt{2})$ $(3,\sqrt{2})$ $(3,-\sqrt{2})$}

For more practice, see Dolciani, pp. 464 - 467.
Then try Trial Run.

ALGEBRA 2

NON-LINEAR SYSTEMS

TRIAL RUN

By inspection determine (1) The type of graph determined by each equation
(2) The maximum number of points of intersection for two such graphs.

1. $y = x^2 + 8$
   $x + 2y = 7$

2. $x^2 + y^2 = 16$
   $xy = 7$

3. $3x = y - 2$
   $3x^2 + 2y^2 = 9$

4. $x^2 - y^2 = 9$
   $x = \frac{2}{y}$

5. $x + 2y^2 = 8$
   $5x^2 + 3y^2 = 30$

6. $x^2 + y^2 = 4$
   $x^2 + y^2 = 1$

7. $x^2 - y^2 = 0$
   $5x^2 + 5y^2 = 45$

8. $3x^2 + 5y^2 = 12$
   $7x^2 + 2y^2 = 14$

II. Solve each of the following systems algebraically:

1. $x^2 + y^2 = 25$
   $x + y^2 = 5$

2. $xy = 6$
   $2y = 3x - 16$

3. $5x^2 - y^2 = 3$
   $x^2 + 2y^2 = 5$

4. $x^2 + y^2 = 8$
   $xy = 4$

5. $4x^2 + 9y^2 = 36$
   $y - x = 3$

6. $x^2 + y^2 = 4$
   $y^2 = x^2$

7. $x^2 + y^2 = 13$
   $2x - y = 4$

8. $4x^2 + 9y^2 = 72$
   $x - y^2 = -1$

9. $x^2 - y^2 = 5$
   $2x - y = 4$

10. $x^2 - y^2 = -1$
    $x^2 + y^2 = 13$

III. Graph each of the following equations:

1. $x^2 + y^2 = 9$
2. $xy = 4$
3. $x = y^2 - 9$
4. $4x^2 + 25y^2 = 100$
5. $x^2 = y^2$
6. $y^2 - x^2 = 9$
7. $y = x^2$
8. $x = y + 4$

ALGEBRA 2

SYSTEMS OF NON-LINEAR EQUATIONS

TRIAL RUN ANSWERS

I. 1. (1) Parabola
Line
(2) 2

2. (1) Circle
Hypertola
(2) 4

3. (1) Line
Ellipse
(2) 2

4. (1) Hyperbola
Hyperbola
(2) 4

5. (1) Parabola
Ellipse
(2) 4

6. (1) Circle
Cir le
(2) 2

7. (1) Intersecting lines
Circle
(2) 4

8. (1) Ellipse
Ellipse
(2) 4

II. 1. $\{(5,0),\ (-4,\pm 3)\}$

2. $\{(6,1),\ (-\frac{2}{3},-9)\}$

3. $\{(1,\ \pm\sqrt{2}),(-1,\ \pm\sqrt{2})\}$

4. $\{(2,2),\ (-2,-2)\}$

5. $\{(-\frac{15}{13},\ \frac{24}{13}),\ (-3,0)\}$

6. $\{(\sqrt{2},\ \pm\sqrt{2}),\ (-\sqrt{2},\ \pm\sqrt{2})\}$

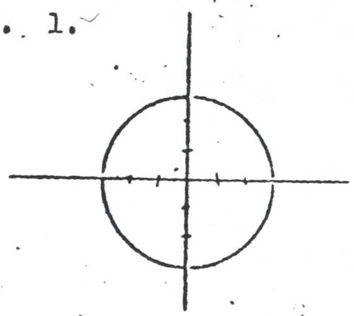
7. $\{(3,2),\ (\frac{1}{5},\ -\frac{18}{5})\}$

8. $\{(3,\ \pm 2)\}$

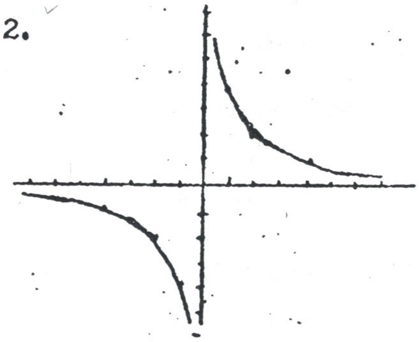
9. $\{(3,2),\ (\frac{7}{3},\ \frac{2}{3})\}$

10. $\{(\sqrt{6},\ \pm\sqrt{7}),\ (-\sqrt{6},\ \pm\sqrt{7})\}$

III. 1.



2.



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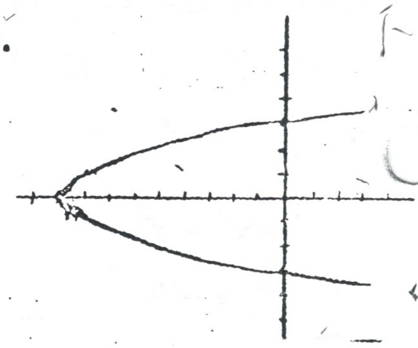
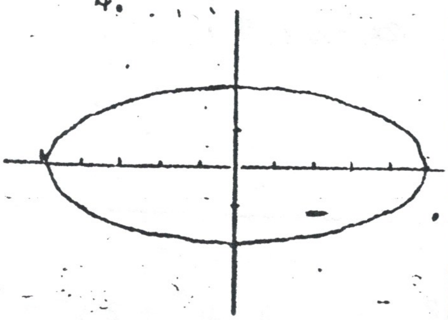


4.



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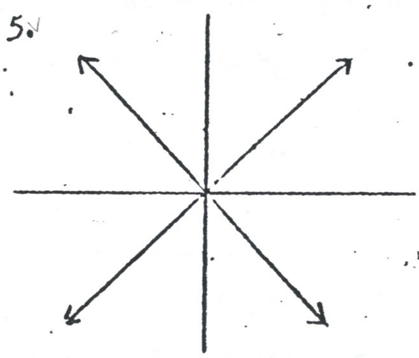

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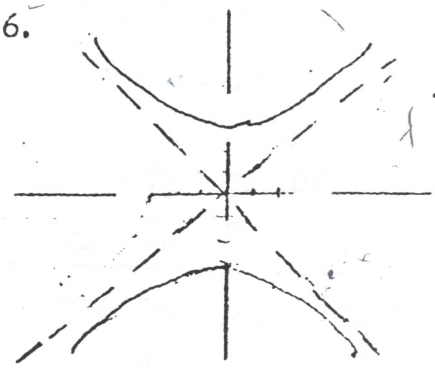

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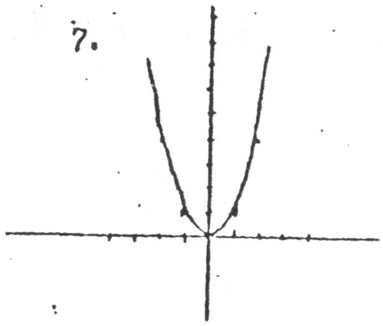

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