Math Analysis/Analytic Geometry

CIRCLE L.A.P.

A circle is a happy thing to be--
Think how the joyful perpendicular
Erected at the kiss of tangency
Must meet my central point, my avatar.
And lovely as I am, yet only three points
Are needed to determine me.
Christopher Morley

BEHAVIORAL OBJECTIVES

I. Given the equation of a circle

A. Determine the center of the circle
B. Determine the length of the radius
C. Graph the circle
D. Determine the area of the circle
E. Determine the circumference of the circle

II. For a given circle find

A. The slope of the tangent to the circle at a given point on the circle
B. The equation of the tangent to the circle at a given point on the circle.
C. The slope of the normal of a circle to a given point on the circle
D. The equation of the normal of a circle to a given point on the circle.

III. Given three non-collinear points find the equation of the circle they determine.

IV. Given the coordinates of the center and the length of the radius

A. Graph the circle
B. Write the equation of the circle

V. Given the center and one point on the circle determine the equation of the circle.

VI. Prove elementary plane geometry theorems for circles using coordinate geometry methods

VII. Define

A. Circle
B. Radius of a circle

VIII. Describe how a plane must intersect a right circular double-napped cone so that the intersection will be a circle.

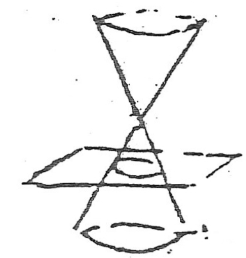
SECTION I CIRCLES

Definition: A circle is the locus of points P which are at a fixed distance from a fixed point called the center. The fixed distance is the radius.

The circumference of a circle is $2\pi r$ units.

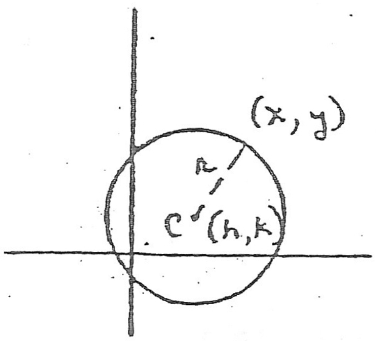
The area of a circle is πr^2 square units.

The circle as a conic section:



The intersection of a plane perpendicular to the axis of a right circular double-napped and the cone is a circle.

Consider the circle on the coordinate plane. For any point (x,y) on the circle, the distance to the center of the circle is constant. This distance is called the radius. Hence we can use the distance formula to derive the equation of the circle.



$$\sqrt{(x-h)^2 + (y-k)^2} = r$$

$$(x-h)^2 + (y-k)^2 = r^2$$

Ah, yes, the above is the equation of the circle with center (h,k) and radius r.

If a circle equation is given in the form $(x-h)^2 + (y-k)^2 = r^2$ it is simple to pick out (h,k) as the center and r as the radius. A circle equation may not always be in this informational form. Sometimes it is in standard form: $x^2 + y^2 + Cx + Dy + E = 0$. In this case it is necessary to complete the square for the x and the y groups to get the equation in the informational form. It is then that one can read off the center coordinates and the radius length.

Example: $16x^2 + 16y^2 - 16x + 8y = 4$

$$16x^2 - 16x + 16y^2 + 8y = 4$$

$$16(x^2 - x) + 16(y^2 + \tfrac{1}{2}y) = 4$$

$$16(x^2 - x + \tfrac{1}{4}) + 16(y^2 + \tfrac{1}{2}y + \tfrac{1}{16}) = 4 + 4 + 1$$

$$16(x - \tfrac{1}{2})^2 + 16(y + \tfrac{1}{4})^2 = 9$$

$$(x - \tfrac{1}{2})^2 + (y + \tfrac{1}{4})^2 = \frac{9}{16}$$

rom the final line we determine that the center of the circle is $(\frac{1}{2}, -\frac{1}{4})$ and ie radius of the circle is 3/4.

ERCISE I GETTING ACQUAINTED WITH THE CIRCLE

Write the equation of each of the following circles:

(a) Center (4,3) and radius 5.

(b) Center (-2,4) and diameter 8.

(c) Center the midpoint of the segment with endpoints (2,10) and (4,18); and diameter 16.

(d) Diameter with endpoints (0,0) and (16,20).

(e) Center (7,6) and the circle contains the point (2,4).

(f) Center (4,5) and is tangent to the line $y = x$.

(g) Center (-1,3) and tangent to the line $3x + 4y = 0$.

(h) Center is the intersection of the lines $3x + 4y = 26$ and $5x - y = 5$; the radius is 4.

(i) Concentric with the circle $(x - 3)^2 + (y - 9)^2 = 5$ and has a radius 7.

(j) Tangent to the lines $x = 7$ and $y = 3$ and has a radius of 2. (4 solutions.)

(k) Has for diameter the portion of $2x - y + 1 = 0$ lying in the second quadrant.

(l) Is tangent to both axes and has its center at (4,4).

(m) Is tangent to both axes and has a radius of 12. (4 solutions.)

(n) Center at (4,5) and circumference 12π.

(o) Center at (-2,3) and area 16π.

(p) Center (0,0) and contains the point (5,12).

2. Find the center and radius of each of the following circles:

(a) $(x - 4)^2 + (y - 3)^2 = 25$

(b) $(x + 3)^2 + (y - 6)^2 = 7$

(c) $(x - 3)^2 + y^2 - 10y + 25 = 9$

(d) $(x - 6)^2 + y^2 - 6y = 40$

(e) $x^2 + 4x + y^2 + 10y = 7$

(f) $2x^2 + 8x + y^2 + 6y = 0$

(g) $4x^2 + 8x + 4y^2 + 16y + 8 = 1$

(h) $5x^2 = -5y^2 - 20y + 2$

(i) $x^2 = 9 - y^2$

(j) $x^2 + y^2 - 10x - 20y = 0$.

SECTION 2 — AND LOVELY AS I AM, YET ONLY THREE POINTS ARE NEEDED TO DETERMINE ME

The standard circle equation is $x^2 + y^2 + Cx + Dy + E = 0$.

Note, in the above equation the constants C, D, and E are needed to fix the circle. If we are given three points (x,y) we can create three equations with three unknowns and solve the system to determine C, D, and E.

Example: Given the three non-collinear points (5,4), (3,2) and (-3,0) write the equation of the circle containing them.

First create the three equations:

$$25 + 16 + 5C + 4D + E = 0 \quad (1)$$

$$9 + 4 + 3C + 2D + E = 0 \quad (2)$$

$$9 + 0 + -3C + 0 + E = 0 \quad (3)$$

The equations on page 3 can be simplified to:

$$5C + 4D + E = -41$$
$$3C + 2D + E = -13$$
$$-3C \quad + E = -9$$

Solve the above system to determine C, D, and E.

You should find this result: C = 6, D = -20, and E = 9

Hence, the circle equation is: $x^2 + y^2 + 6x - 20y + 9 = 0$.

If three points are collinear they do not determine a circle.

Since three points determine a circle it follows that any triangle can be inscribed in a circle.

Recall from geometry that if a right triangle is inscribed in a circle it is inscribed in a semi-circle and the hypotenuse is the diameter of the circle.

Solving a system of three equations in three unknowns can be rough. At times the solution can be found in other ways. Do not forget to use your imagination and try for quick solutions.

XERCISE 2

. From Protter page 97. Find the eqations of the circles through the given points:

(a) (-5,2), (-3,4), (1,2) (b) (-3,1), (-2,2), (6,-2)

(c) (7,-2), (3,-4), (0,3) (d) (1,-1), (0,1), (-3,-2)

. Find the center and radius of the circle which contains each of the following ... of points:

(a) (-1,0), (2,0), (-1,-4) (b) (2,-1), (-2,-1), (5,0)

Find the equation of the circle circumscribed about the right triangle with endpoints of the hypotenuse

(a) (2,4) and (6,10) (b) (0,0) and (14,2)

Find the equation of the circle circumscribed about the triangle

(a) Formed by the intersection of the lines $x - y = 4$, $x + 2y = 10$, and $3x + y = 0$.

(b) The X-axis, the Y-axis and the line $5x - 3y = 8$.

(c) The lines $x = y$, $x = -y$ and $x = 6$.

(d) The lines X-axis, $y = x$ and the line $x = 10$.

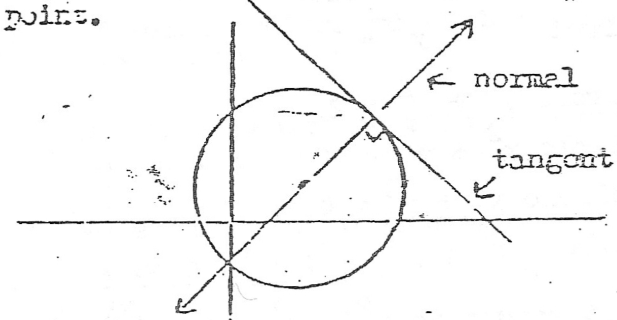
(e) The lines $x = 8$, $y = 10$ and the line $3x + 4y = 10$

5. Write the equation of the circle circumscribed around the trapezoid with vertices (0,0), (2,6), (4,6) and (6,0).

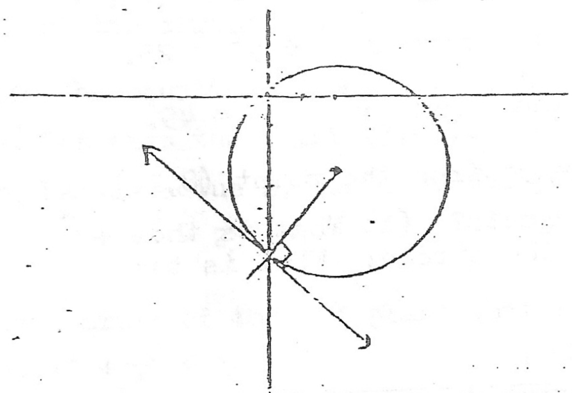
6. Find the equations of the two circles which are tangent to the circle $x^2 + y^2 = 169$, contain the point (5,12), and have a radius of 5. (Hint, set up pairs of similar triangles and use proportion.)

SECTION III. TANGENTS AND NORMALS

At any point on a circle a tangent may be drawn. The tangent is perpendicular to the radius drawn to the point of tangency. The line containing the radius is called the normal to that point.



For the circle: $(x - 2)^2 + (y + 3)^2 = 8$.



Consider the point (0,-5) on the circle.

The slope of the normal is 1.

The equation of the normal is

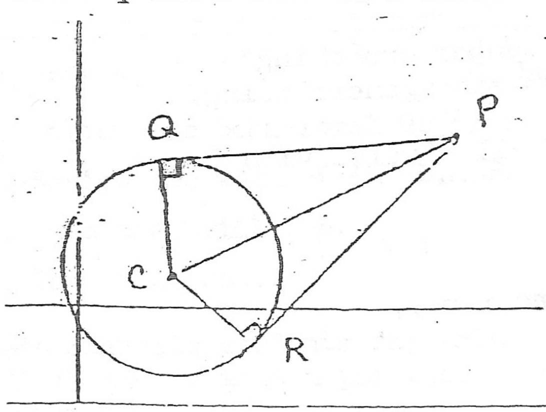
$y = x - 5.$

The slope of the tangent is -1.

The equation of the tangent is

$y = -x - 5.$

From a point P outside a circle two tangents can be drawn to the circle.



Since CQ = CR, and CP = CP, and both triangles are right triangles, it follows that PQ = PR. In other words, the tangent segments are congruent.

It can easily be shown using the Pythagorean theorem that:

$$PQ^2 = CP^2 - CQ^2$$

In the following exercises you will be asked to work problems dealing with tangents, normals, and the length of the tangent segments. Be sure that you understand the concepts touched upon above.

EXERCISE 3

1. For each of the following find the equations of the lines tangent to the given circle and normal to the given circle at the point P on the circle:

   (a) $x^2 + y^2 + 2x - 4y = 0$; P: (1,3)

   (b) $(x - 5)^2 + (y - 1)^2 = 25$; P: (0,1)

   (c) $x^2 + y^2 + 2x - 19 = 0$; P: (-3,4)

   (d) $x^2 + y^2 + 8x + 2y + 16 = 0$; P: (-4,-2)

2. Find the length of the tangent segment from

   (a) point (4,7) to the circle $x^2 + y^2 = 7$

   (b) point (9,9) to the circle $x^2 + y^2 = 2$

   (c) point (6,1) to the circle $x^2 + y^2 = 37$.

   (d) point (1,-1) to the circle $x^2 + y^2 = 4$.

   (e) point (10,1) to the circle $x^2 + y^2 - 6x - 8y = 0$.

3. Find the equations of the tangents to $x^2 + y^2 = 36$ from the point (0,12). What is the measure of the angle between the two tangents?

. Find the equation of the two horizontal tangents to the circle $x^2 + y^2 = 25$.

. Find the equations of the two vertical tangents to the circle $x^2 + y^2 = 49$.

. Two tangents are drawn to the circle $x^2 + (y - 5)^2 = 36$ from the point (8,10). What is the measure of the angle between the two tangents? (b) What is the length of the tangent segment? What is the area of the circle? What is the circumference of the circle?

. Find the area and circumference for each circle in #2.

CTION IV. A DEEPER LOOK AT CIRCLES

st of the problems in the previous sections were of the "number crunching" pe. In this section we shall do some more of that plus do some generalizing.

r example, suppose we would want to write the equation of all circles with nter (5,3).

lution: $(x - 5)^2 + (y - 3)^2 = r^2$ for $r > 0$.

w about the equation of all circles with center on the line x = y.

lution: $(x - h)^2 + (y - h)^2 = r^2$ for $r > 0$.

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ptional for Analytic Geometry students.

Try your hand at the following:

EXERCISE 4

1. Write the equation for the family of circles which
   (a) Lie in the first quadrant and are tangent to the X and Y axes.
   (b) Are tangent to the Y-axis and the line $x = 6$.
   (c) Are tangent to the X-axis and the line $y = 10$.
   (d) Are concentric with the circle $(x - 3)^2 + y^2 = 25$.
   (e) Are externally tangent to the circle $x^2 + y^2 = 4$ and have a radius 7.

2. Write the equation of the set of points which are the centers of all circles which contain the points (4,5) and (3,2).

3. Write the equation of the set of points which contain the centers of all circles externally tangent to the circle $x^2 + y^2 = 9$ and have a radius of 5.

4. Use coordinate geometry to prove each of the following:
   (a) An angle inscribed in a semi-circle is a right angle.
   (b) A radius drawn perpendicular to a chord of a circle bisects the chord.
   (c) The perpendicular bisector of a chord of a circle contains the center of the circle.

5. Find the equation of the circle which
   (a) Contains the point (3,-2), has center on $2x - y + 2 = 0$, and has a radius 5.
   (b) Has a radius 2, contains the point (3,4) and is tangent to the circle $x^2 + y^2 = 25$.

6. Shade areas of the X-Y plane that represent the graphs of the following:
   (a) $x^2 + y^2 \leq 16$ (b) $x^2 + y^2 \leq 16 \cap y \leq x$
   (c) $x^2 + y^2 \leq 16 \cap x - y \leq 0 \cap x + y \geq 4$.

SECTION V. EVALUATION

1. Review the Behavioral Objectives of the L.A.P.
2. Review the work of the L.A.P.
3. Take the Trial Run.
4. Take the test.

Math Analysis students may take a brief oral test on the major concepts of this L.A.P. and present a paper on some circle topic.

Some suggestions: 1. The geometry of the shpere. 2. The number π.

3. The historical development of the formulas for area and circumference of a circle.

Answers More or Less!

Exercise 1

1. (a) $(x - 4)^2 + (y - 3)^2 = 25$
   (b) $(x + 2)^2 + (y - 4)^2 = 16$
   (c) $(x - 3)^2 + (y - 14)^2 = 64$
   (d) $(x - 8)^2 + (y - 10)^2 = 164$
   (e) $(x - 7)^2 + (y - 6)^2 = 29$
   (f) $(x - 4)^2 + (y - 5)^2 = \frac{1}{2}$
   (g) $(x + 1)^2 + (y - 3)^2 = 81/25$
   (h) $(x - 2)^2 + (y - 1)^2 = 16$
   (i) $(x - 3)^2 + (y - 9)^2 = 49$
   (j) $(x - 5)^2 + (y - 5)^2 = 4$
   $(x - 5)^2 + (y - 1)^2 = 4$
   $(x - 9)^2 + (y - 1)^2 = 4$
   $(x - 9)^2 + (y - 5)^2 = 4$
   (k) $(x + \frac{1}{4})^2 + (y - \frac{1}{2})^2 = 5/16$
   (l) $(x - 4)^2 + (y - 4)^2 = 16$
   (m) $(x - 12)^2 + (y \pm 12)^2 = 144$
   $(x + 12)^2 + (y \pm 12)^2 = 144$
   (n) $(x - 4)^2 + (y - 5)^2 = 36$
   (o) $(x + 2)^2 + (y - 3)^2 = 16$
   (p) $x^2 + y^2 = 169$

2. (a) $(4,3)$; 5 (b) $(-3,6)$; $\sqrt{7}$ (c) $(3,5)$; 3 (d) $(6,3)$; 7
   (e) $(-2,-5)$; 6 (f) not a circle; (g) $(-1,-2)$; $\sqrt{13}/2$ (h) $(0,2)$; $\sqrt{\frac{22}{5}}$
   (i) $(0,0)$; 3 (j) $(5,10)$; $\sqrt{125}$

Exercise 2

1. (a) $x^2 + y^2 + 4x - 2y - 5 = 0$ (b) $x^2 + y^2 - 2x + 4y - 20 = 0$
   (c) $17x^2 + 17y^2 - 114x - 10y - 123 = 0$ (d) $3x^2 + 3y^2 + 7x + 5y - 8 = 0$

2. (a) $(\frac{1}{2},-2)$; $\frac{5}{2}$ (b) $(0,10)$; $\sqrt{125}$

3. $(x - 4)^2 + (y - 7)^2 = 13$ (b) $(x - 7)^2 + (y - 1)^2 = 50$

4. (a) $x^2 + y^2 - 2x - 4y - 20 = 0$ (b) $(x - \frac{4}{5})^2 + (y - \frac{3}{5})^2 = \frac{544}{225}$
   (c) $(x - 6)^2 + y^2 = 36$ (d) $(x - 5)^2 + (y - 5)^2 = 50$
   (e) $(x + 1)^2 + (y - \frac{13}{4})^2 = \frac{2025}{16}$

5. $3x^2 + 3y^2 - 18x - 14y = 0$

6. $(x - \frac{90}{13})^2 + (y - \frac{216}{13})^2 = 25$; $(x - \frac{40}{13})^2 + (y - \frac{96}{13})^2 = 25$

Answers Exercise 3

1. (a) T: $y = -2x + 5$; N: $2y = x + 5$ (b) T: $x = 0$; N: $y = 1$
   (c) T: $2y = x + 11$; N: $y = -2x - 2$ (d) T: $= 2$; N: $= -4$

2. (a) $\sqrt{58}$; (b) $4\sqrt{10}$; (c) 0, (6,1) is on the circle.
   (d) no tangent segment, (1,-1) is in the interior of the circle. (e) $\sqrt{33}$.

3. 60° 4. $y = \pm 5$; 5. $x = \pm 7$. 6. (a) 79°; (b) $\sqrt{53}$; (c) 36π; (d) 12π

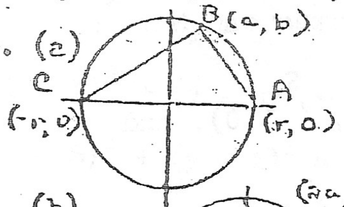
7. (a) 7π; $2\sqrt{7}\,\pi$; (b) 2π $2\sqrt{2}\,\pi$; (c) 37π, $2\sqrt{37}\,\pi$;
   (d) 4π; 4π (e) 25π, 10π

Exercise 4

1. (a) $(x - h)^2 + (y - h)^2 = h^2$ for $h > 0$
   (b) $(x - 3)^2 + (y - k)^2 = 9$; (c) $(x - h)^2 + (y - 5)^2 = 25$
   (d) $(x - 3)^2 + y^2 = r^2$ for $r > 0$ and $r \neq 5$.
   (e) $(x - h)^2 + (y - k)^2 = 49$ for $h^2 + k^2 = 81$

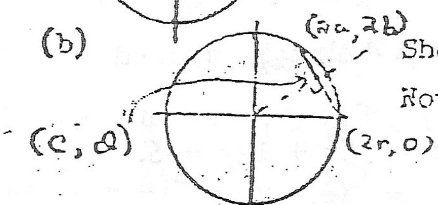
2. $3y + x = 14$. 3. $x^2 + y^2 = 64$

4. (a)



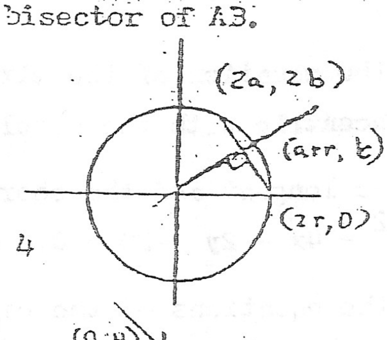
Show slope of $\overleftrightarrow{AB}$ · slope of $\overleftrightarrow{CB} = -1$
Note: $a^2 + b^2 = r^2$

(b)



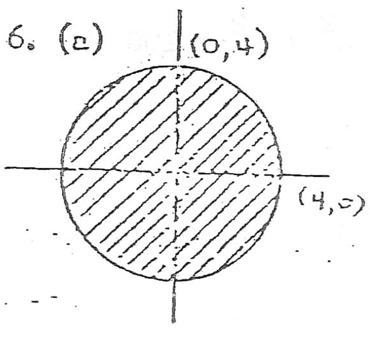
Show $c = a + r$ and $d = b$
Note: $a^2 + b^2 = r^2$

(c) Show (0,0) is on the perpendicular bisector of AB.

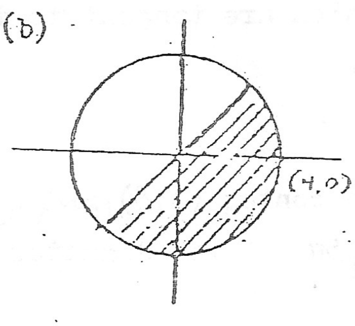


5. (a) $(x - 2)^2 + (y - 2)^2 = 25$ and $(x + 2)^2 + (y + 2)^2 = 25$
   (b) $(x - \frac{21}{5})^2 + (y - \frac{28}{5})^2 = 4$; $(x - \frac{9}{5})^2 + (y - \frac{12}{5})^2 = 4$

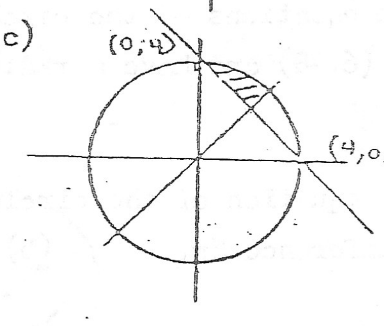
6. (a)



(b)



(c)



MATH ANALYSIS/ANALYTIC GEOMETRY CIRCLES TRIAL RUN

Zoom through this Trial Run. One of our problems seems to be lack of speed. Become an expert!

1. Define: (a) Circle (b). Radius of a circle.

2. Find the center, radius, area, and circumference of each of the following circles:

   (a) $(x - 3)^2 + (y + 2)^2 = 16$ (b) $x^2 + y^2 - 10x + 2y + 22 = 0$

   (c) $x^2 + y^2 - 25 = 0$

3. For the circle $(x - 3)^2 + (y - 5)^2 = 25$ find

   (a) The equation of the normal to the circle at (6,9).

   (b) The equation of the tangent to the circle at (6,9).

   (c) The length of the tangent segment from the point (12,12) to the circle.

   (d) The measure of the angle between the two tangents from (12,12) to the circle.

4. If (7,3) is the center of a circle and (0,0) is on the circle, the equation of the circle is ____________________.

5. What is the equation of a circle which is concentric with the circle whose equation is $(x - 4)^2 + (y + 3)^2 = 20$ and has a radius of 2?

6. What is the equation of a circle which is tangent to the X and Y axes and has its center at (-3,3)?

7. Write the equation of the circle which contains the points (2,-1), (-3,0), and (1,4).

8. Find the equation of the circle which is circumscribed around a triangle formed by the lines $x = 0$, $y = 0$, and $y = -3x + 6$.

9. Find the equation of the circle with center (4,8) and tangent to $5x + 12y = 6$.

10. Find the equation of the circle whose diameter is the segment from (4,2) to (8,10).

11. Write the equation of the circle which is tangent to the line $3x - 4y + 17 = 0$ and concentric with the circle $x^2 + y^2 - 4x + 6y - 10 = 0$.

12. Find the longest and the shortest distance from (10,7) to the circle $x^2 + y^2 - 4x - 2y - 20 = 0$.

13. Write the equations of the circles which are tangent to the circle $x^2 + y^2 = 100$ at point (6,-8) and have a radius 5.

14. Write the equation of the circle with center (4,6) and

   (a) circumference 8π (b) area 5π (c) diameter 3

Math Analysis Students only:

15. Write the equation of the set of points which

   (a) Are the centers of all circles with radius 5 and tangent to the circle $(x - 4)^2 + (y + 3)^2 = 9$.

   (b) Are centers of the circles which are tangent to the lines $y = \pm 7$.

   (c) Are the centers of the circles which contain the points (0,0) and (4,8).

   (d) Are the centers of the circles which are tangent to the X-axis and the line containing (0,0) with an inclination of 60°.

16. Review, very carefully, the final section of the L.A.P.

ANSWERS

1. (a) the set of all points in a plane at a given distance from a fixed point called the center.
   (b) the distance from the center of a circle to a point on the circle.

2. (a) (3,-2); 4; 16π; 8π.   (b) (5,-1); 2; 4π; 4π.   (c) (0,0); 5; 25π; 10π.

3. (a) $3y = 4x + 3$;   (b) $4y = -3x + 54$;   (c) $\sqrt{105}$;   (d) 52°.

4. $(x - 7)^2 + (y - 3)^2 = 58$.   5. $(x - 4)^2 + (y + 3)^2 = 4$

6. $(x + 3)^2 + (y - 3)^2 = 9$ 7. $3x^2 + 3y^2 + x - 7y - 24 = 0$

8. $(x - 1)^2 + (y - 3)^2 = 10$ 9. $(x - 4)^2 + (y - 8)^2 = \left(\frac{110}{13}\right)^2$

10. $(x - 6)^2 + (y - 6)^2 = 20$ 11. $(x - 2)^2 + (y + 3)^2 = 49$

12. 5, 15.   13. $(x - 9)^2 + (y - 12)^2 = 25$ and $(x - 3)^2 + (y - 4)^2 = 25$

14. (a) $(x - 4)^2 + (y - 6)^2 = 16$ (b) $(x - 4)^2 + (y - 6)^2 = 5$
   (c) $(x - 4)^2 + (y - 6)^2 = \frac{9}{4}$.

15. (a) $(x - 4)^2 + (y - 3)^2 = 64$;   (b) $y = 0$
   (c) $2y = -x + 10$ (d) $\sqrt{3}\,y = x$

Yes, a circle is a happy thing to be!