Math Analysis CONDITIONAL EQUATIONS

Trigonometry LEARNING ACTIVITIES PACKAGE

Suggested time: 1.5 weeks

This is another L.L. (Little L.A.P.). Again it will take C.P. (Concentrated Practice.)

Behavioral Objectives

1. Solve conditional trig equations over the domain $0 \leq \theta \leq 2\pi$

* 2. Solve conditional trig equations over all defined domain of the function.
  * Not required of trig students.

SECTION I. CONDITIONAL EQUATIONS OVER DOMAIN $0 \leq \theta \leq 2\pi$

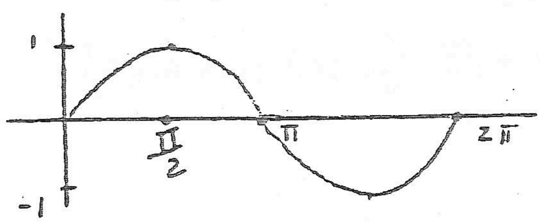
This L.A.P. can best be explained be giving some examples. Study each of the following carefully.

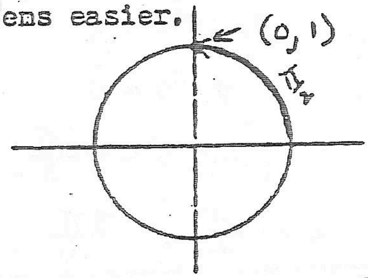
Example 1: $\sin\theta - 1 = 0$

$$\sin\theta = 1$$

$$\theta = \frac{\pi}{2}$$

In line 2, it is determined that $\sin\theta = 1$. Then we question for what θ does $\sin\theta = 1$? The solution can be obtained by considering the unit circle or the graph of the sin function, which ever seems easier.





$\frac{\pi}{2}$ is the only solution for this equation because $\sin\theta = 1$ only at $\frac{\pi}{2}$.

Example 2: $\cos 2\theta - \sin\theta = 0$

$$1 - 2\sin^2\theta - \sin\theta = 0$$

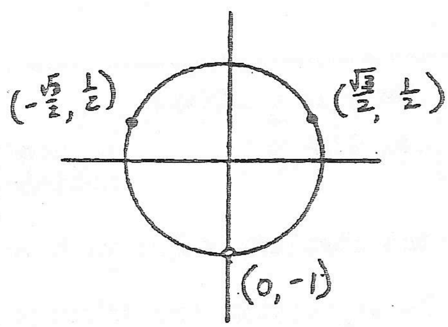
$$2\sin^2\theta + \sin\theta - 1 = 0$$

$$(2\sin\theta - 1)(\sin\theta + 1) = 0$$

$$2\sin\theta - 1 = 0 \quad \text{or} \quad \sin\theta + 1 = 0$$

$$\sin\theta = \tfrac{1}{2} \quad \text{or} \quad \sin\theta = -1$$

Solution: $\theta = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{3\pi}{2}$



Some comments: 1. Study the algebra used above. 2. The substitution choice for $\cos 2\theta$ was made with the hope to write the equation using one function only. 3. Notice how the solution points are symmetric on the unit circle.

Example 3: $2\sin\theta\cos\theta = -\sin\theta$

$$2\sin\theta\cos\theta + \sin\theta = 0$$

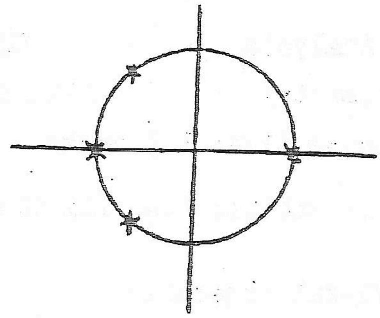
$$\sin\theta\,(2\cos\theta + 1) = 0$$

$$\sin\theta = 0 \quad \text{or} \quad 2\cos\theta + 1 = 0$$

$$\theta = 0, \pi, 2\pi \quad \text{or} \quad 2\cos\theta = -1$$

$$\cos\theta = -\frac{1}{2}$$

$$\theta = \frac{2\pi}{3}, \frac{4\pi}{3}$$



Solution: $\theta = 0, \pi, 2\pi, \frac{2\pi}{3}, \frac{4\pi}{3}$

Comment: In line 3 do not be tempted to divide by sin θ. Lots of folks do this and lose solutions. Again notice the symmetry of the solutions when they are placed on the unit circle.

Example 4: $2\sin^4\theta - 5\sin^2\theta = -2$

$$2\sin^4\theta - 5\sin^2\theta + 2 = 0$$

$$(2\sin^2\theta - 1)(\sin^2\theta - 2) = 0$$

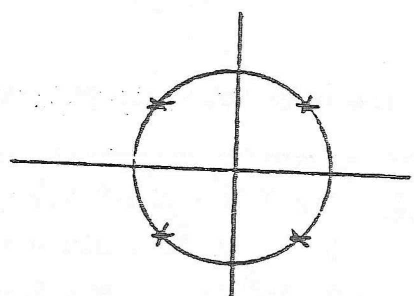
$$2\sin^2\theta - 1 = 0 \qquad \sin^2\theta - 2 = 0$$

$$2\sin^2\theta = 1 \qquad \sin^2\theta = 2$$

$$\sin^2\theta = \frac{1}{2} \qquad \sin\theta = \pm\sqrt{2} \quad [\text{reject } |\sin\theta| \le 1]$$

$$\sin\theta = \pm\frac{\sqrt{2}}{2}$$

Solution: $\theta = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$

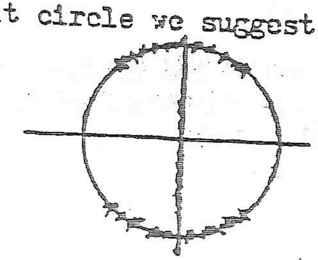


Example 5 Here is a nifty!

$|\cos\theta| \le \frac{1}{2}$ We don't need a lot of algebra here, just a good notion of the unit circle...and head power.

Removing the absolute value we have: $-\frac{1}{2} \le \cos\theta \le \frac{1}{2}$. Observing the unit circle we suggest the solution:

$$\frac{\pi}{3} \le \theta \le \frac{2\pi}{3} \quad \text{or} \quad \frac{4\pi}{3} \le \theta \le \frac{5\pi}{3}$$



Example 6: $\cos 5\theta + \cos 3\theta = 0$

$$\cos(4\theta + \theta) + \cos(4\theta - \theta) = 0$$

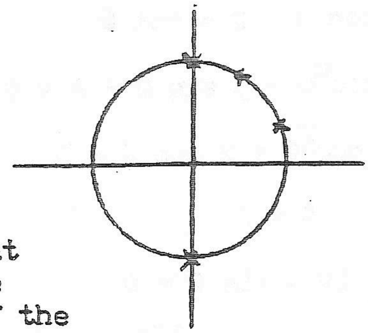
$$\cos 4\theta \cos \theta - \sin 4\theta \sin \theta + \cos 4\theta \cos \theta - \sin 4\theta \sin \theta = 0$$

$$2 \cos 4\theta \cos \theta = 0$$

$$\cos 4\theta = 0 \quad \text{or} \quad \cos \theta = 0$$

$$4\theta = \frac{\pi}{2}, \ \frac{3\pi}{2} \qquad \theta = \frac{\pi}{2}, \ \frac{3\pi}{2}$$

$$\theta = \frac{\pi}{8}, \ \frac{3\pi}{8}$$



When the solutions projected above are put on the unit circle we notice a glaring lack of symmetry. That make us suspicious that there are more solutions. The 4θ is the trouble maker. Using the periodic nature of the cos function we have:

$$\cos 4\theta = 0$$

$$4\theta = \frac{\pi}{2} + 2\pi \qquad 4\theta = \frac{3\pi}{2} + 2\pi$$

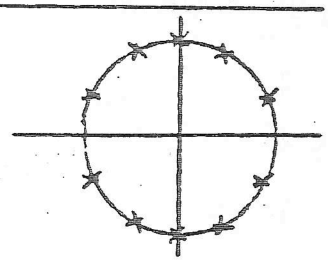
$$\theta = \frac{\pi}{8} + \frac{\pi}{2} \qquad \theta = \frac{3\pi}{8} + \frac{\pi}{2}$$

So, two more solutions. By adding 4π, and then 6π, we can get four more solutions.

The complete solution set is:

$$\frac{\pi}{8}, \ \frac{3\pi}{8}, \ \frac{5\pi}{8}, \ \frac{7\pi}{8}, \ \frac{9\pi}{8}, \ \frac{11\pi}{8}, \ \frac{13\pi}{8}, \ \frac{15\pi}{8}, \ \frac{\pi}{2}, \ \frac{3\pi}{2}$$

And look at that symmetry!



Example 7: The last one.

$$\frac{1 - \sin\theta}{\cos\theta} = \cos\theta$$

$$1 - \sin\theta = \cos^2\theta$$

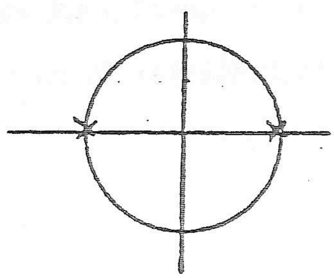
$$1 - \sin\theta = 1 - \sin^2\theta$$

$$\sin^2\theta - \sin\theta = 0$$

$$\sin\theta(\sin\theta - 1) = 0$$

$$\sin\theta = 0 \qquad \sin\theta = 1$$

$$\theta = 0, \pi, 2\pi, \qquad \theta = \frac{\pi}{2}$$



Solution: $0, \pi, 2\pi$

[Note: $\frac{\pi}{2}$ must be rejected because $\cos\frac{\pi}{2} = 0$ and in the original expression we divide by $\cos\theta$.]

Assignment: Solve each of the following equations. Give all solutions for $0 \leq \theta < 2\pi$.

1 $2 \sin \theta - 1 = 0$

2. $4 \cos^2\theta - 3 = 0$

3. $3 \tan^2 \theta - 1 = 0$

4. $\sin^2\theta - \cos^2\theta + 1 = 0$

5. $2 \cos^2\theta - \sqrt{3} \cos \theta = 0$

6. $\sec^2\theta - 4 \sec \theta + 4 = 0$

7. $3 \sec \theta + 2 = \cos \theta$

8. $2 \sin^2\theta - \sin \theta = 0$

9. $2 \sin^2\theta - 5 \sin \theta + 2 = 0$

10. $2 \sin \theta \cos \theta + \sin \theta = 0$

11. $\sqrt{3} \csc^2\theta + 2 \csc \theta = 0$

12. $2 \sin^2\theta + 3 \cos \theta - 3 = 0$

13. $\cos 2 \theta = 0$

14. $4 \tan^2\theta - 3 \sec^2\theta = 0$

15. $\cos 2\theta - \sin \theta = 0$

16. $2 \cos^2\theta + 2 \cos 2\theta = 1$

17. $\cos 2\theta + 2 \cos^2 \frac{\theta}{2} = 1$

18. $\sec^2\theta - 2 \tan \theta = 0$

19. $\cos 2\theta - \cos \theta = 0$

20. $\cos 2\theta \cos \theta + \sin 2\theta \sin \theta = 1$

21. $\cos^2\theta - \sin^2\theta = \sin \theta$

22. $\cos \theta = \frac{1 + \cos^2\theta}{2}$

23. $1 - \sin^2\theta = \cos \theta$

24. $1 + \sin^2\theta = \cos \theta$

25. $\sin 2\theta - \sin \theta = 0$

26. $\cos 3\theta - \cos \theta = 0$

27. $2 \cos^2 2\theta - 2 \sin^2 2\theta = 1$

28. $2 \cos^2\theta - \sin \theta - 1 = 0$

29 $\frac{1 - \cos \theta}{\sin \theta} = \sin \theta$

30. $\cot^2\theta + \csc \theta = 1$

31. $\tan^2\theta = 1$

32. $|\tan \theta| \leq 1$

33. $\cos \theta \leq \frac{1}{2}$

34. $\sin \theta \geq 0$

35 $|\sec \theta| \leq 1$

36. $|\sin \theta| \geq \frac{1}{2}$

Trig students: 1. Check your work; 2. Take the trial run. 3. Take the test.

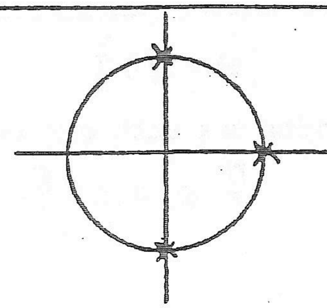
Math Analysis Students: Go on to Section 2.

SECTION 2 CONDITIONAL EQUATIONS OVER ALL DEFINED DOMAIN

This section is much like the other. We still need to find the solution set for the domain $0 \le \theta \le 2\pi$. Then we generalize to give the complete solution. It is important to remember here that the period of sin, cos, sec, and csc is 2π, and the period of tan and cot is π.

As in section 1, the work of this section can best be explained by some examples.

Example 1: Solve for all θ:



$$\cos^2\theta = \cos\theta$$

$$\cos^2\theta - \cos\theta = 0$$

$$\cos\theta(\cos\theta - 1) = 0$$

$$\cos\theta = 0 \qquad \cos\theta = 1$$

$$\theta = \frac{\pi}{2}, \frac{3\pi}{2} \qquad \theta = 0$$

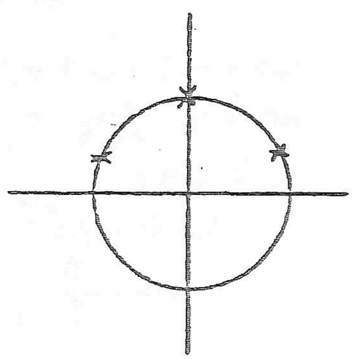
We can generalize in two ways: 1) Using the periodic nature the solutions are: $\left\{\frac{\pi}{2} + 2k\pi, \frac{3\pi}{2} + 2k\pi, 2k\pi.\right\}$

2. Or, more elegantly, observe the unit circle placement of the primary solutions and write: $\left\{\frac{(2k+1)\pi}{2}, 2k\pi\right\}$.

YOU CAN ALWAYS GENERALIZE A SOLUTION BY TAKING THE PRIMARY SOLUTIONS AND ADDING $2k\pi$ or $k\pi$, WHICHEVER IS APPROPRIATE.

However, the solutions presented in this L.A.P. will be in the elegant form.

Example 2: Solve for all θ



$$\cot^2\theta - 3\csc\theta + 3 = 0$$

$$\frac{\cos^2\theta}{\sin^2\theta} - \frac{3}{\sin\theta} + 3 = 0$$

$$\cos^2\theta - 3\sin\theta + 3\sin^2\theta = 0$$

$$1 - \sin^2\theta - 3\sin\theta + 3\sin^2\theta = 0$$

$$1 - 3\sin\theta + 2\sin^2\theta = 0$$

$$(1 - 2\sin\theta)(1 - \sin\theta) = 0$$

$$\sin\theta = \tfrac{1}{2} \qquad \sin\theta = 1$$

$$\theta = \frac{\pi}{6}, \frac{5\pi}{6}, \qquad \theta = \frac{\pi}{2}$$

Try writing a general solution before you turn the page.

Solution: $\theta = \frac{\pi}{6} + 2k\pi = \frac{\pi + 12k\pi}{6} = \frac{(12k + 1)\pi}{6}$

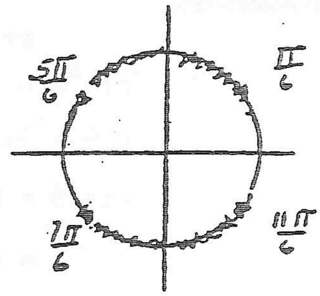
$\theta = \frac{5\pi}{6} + 2k\pi = \frac{5\pi + 12k\pi}{6} = \frac{(12k + 5)\pi}{6}$

$\theta = \frac{\pi}{2} + 2k\pi = \frac{\pi + 4k\pi}{2} = \frac{(4k + 1)\pi}{2}$

Or: Solution: $\frac{(4k + 1)\pi}{2}$, $\frac{(4k + 1)\pi}{2} \pm \frac{\pi}{3}$

Example 3: $|\sin\theta| \geq \frac{1}{2}$

Observing the unit circle we have the primary solution:



$\frac{\pi}{6} \leq \theta \leq \frac{5\pi}{6}$ or $\frac{7\pi}{6} \leq \theta \leq \frac{11\pi}{6}$

Three acceptable statements of the complete solution set:

1. $\frac{\pi}{6} + 2k\pi \leq \theta \leq \frac{5\pi}{6} + 2k\pi$ or $\frac{7\pi}{6} + 2k\pi \leq \theta \leq \frac{11\pi}{6} + 2k\pi$

2. $\frac{(12k + 1)\pi}{6} \leq \theta \leq \frac{(12k + 5)\pi}{6}$ or $\frac{(12 + 7)\pi}{6} \leq \theta \leq \frac{(12k + 11)\pi}{6}$

3. This third solution is by far the most elegant. Notice that the solution set repeats every π units.

   It begins at $\frac{\pi}{6}$, $\frac{7\pi}{6}$, $\frac{13\pi}{6}$, $\frac{19\pi}{6}$ ... $\frac{(6k + 1)\pi}{6}$

   It ends at $\frac{5\pi}{6}$, $\frac{11\pi}{6}$, $\frac{17\pi}{6}$, $\frac{23\pi}{6}$, ... $\frac{(6k + 5)\pi}{6}$

   So, the solution set is: $\frac{(6k + 1)\pi}{6} \leq \theta \leq \frac{(6k + 5)\pi}{6}$

Example 4:

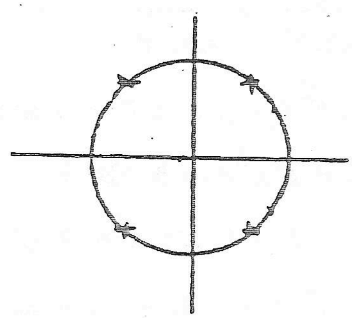
$\tan\theta - 3\cot\theta = 0$

$\tan\theta - \frac{3}{\tan\theta} = 0$

$\tan^2\theta - 3 = 0$

$\tan\theta = \pm\sqrt{3}$

$\theta = \frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}$



The general solution: $\theta = k\pi \pm \frac{\pi}{3}$

Assignment 2

1. Refer to assignment 1 on page 4. Generalize the solutions for #1 through # 10, and # 29 through 36.

2. Solve each of the following for all values of θ:

(a) $2 - \sin\theta = 2\cos^2\theta$ (b) $2\sin^2\theta - \sin\theta - 1 = 0$

(c) $2\sec\theta + 4 = 0$ (d) $\sin\theta + \cos\theta\tan\theta = -2$

(e) $\sin^2\theta - 2\sin\theta + 1 = 0$ (f) $3\cot^2\theta - 1 = 0$

(g) $2\cos^2\theta - \cos\theta = 0$ (h) $\tan^2\theta - 2\tan\theta + 1 = 0$

(i) $\tan\theta - 3\cot\theta = 0$ (j) $\cos 2\theta + \sin 2\theta = 0$

Review your work. Take the Trial Run. Take the test.

Answers:

Assignemt 1:

1. $\frac{\pi}{6}, \frac{5\pi}{6}$; 2, 3. $\frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}$; 4. $0, \pi, 2\pi$;

5. $\frac{\pi}{2}, \frac{3\pi}{2}, \frac{\pi}{6}, \frac{11\pi}{6}$; 6. $\frac{\pi}{3}, \frac{5\pi}{3}$; 7. π; 8. $0; \pi, 2\pi, \frac{\pi}{6}, \frac{5\pi}{6}$,

; 9. $\frac{\pi}{6}, \frac{5\pi}{6}$; 10. $0, \pi, 2\pi, \frac{2\pi}{3}, \frac{4\pi}{3}$; 11. $\frac{4\pi}{3}, \frac{5\pi}{3}$;

12. $0, 2\pi, \frac{\pi}{3}, \frac{5\pi}{3}$; 13. $\frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$; 14. $\frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}$;

15. $\frac{3\pi}{2}, \frac{\pi}{6}, \frac{5\pi}{6}$; 16. $\frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$; 17. $\frac{\pi}{3}, \frac{5\pi}{3}, \pi$

18. $\frac{\pi}{4}, \frac{5\pi}{4}$; 19. $0, 2\pi, \frac{2\pi}{3}, \frac{4\pi}{3}$; 20. $0, 2\pi$. 21. $\frac{3\pi}{2}, \frac{\pi}{6}, \frac{5\pi}{6}$;

22. $0, 2\pi$; 23. $\frac{\pi}{2}, \frac{3\pi}{2}, 0, 2\pi$; 24. $0, \ , 2\pi$; 25. $\frac{\pi}{3}, \frac{5\pi}{3}, 0, \pi, 2\pi$

26. $0, \pi, 2\pi, \frac{\pi}{2}, \frac{3\pi}{2}$; 27. $\frac{\pi}{12}, \frac{7\pi}{12}, \frac{13\pi}{12}, \frac{19\pi}{12}, \frac{5\pi}{12}, \frac{11\pi}{12}, \frac{17\pi}{12}, \frac{23\pi}{12}$;

28. $\frac{\pi}{6}, \frac{5\pi}{6}, \frac{3\pi}{2}$; 29. $\frac{\pi}{2}, \frac{3\pi}{2}$; 30. $\frac{\pi}{2}, \frac{7\pi}{6}, \frac{11\pi}{6}$;

31. $\frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$; 32. $0 \le \theta \le \frac{\pi}{4} \cup \frac{3\pi}{4} \le \theta \le \frac{5\pi}{4} \cup \frac{7\pi}{4} \le \theta \le 2\pi$;

33. $\frac{\pi}{3} \le \theta \le \frac{5\pi}{3}$; 34. $0 \le \theta \le \pi$; 35. $0, 2\pi, \pi$ 36. $\frac{\pi}{6} \le \theta \le \frac{5\pi}{6} \cup$

$\frac{7\pi}{6} \le \theta \le \frac{11\pi}{6}$

The management went crazy typing these answers. If any are incorrect...speak gently!

Exercises

1. 1. $\frac{(4k+?)\pi}{2} \pm \frac{\pi}{3}$; 2, 3, $k\pi \pm \frac{\pi}{6}$; 4. $k\pi$;

5. $\frac{(2'+1)\pi}{2}$ or $2k\pi \pm \frac{\pi}{6}$; 6. $2k\pi \pm \frac{\pi}{3}$; 7. $(2k+1)\pi$;

8. $k\pi$, or $\frac{(4k+1)\pi}{2} \pm \frac{\pi}{3}$; 9. $\frac{(4k+1)\pi}{2} \pm \frac{\pi}{3}$

10. $k\pi$, or $(2k+1)\pi \pm \frac{\pi}{3}$.

29. $\frac{(2k+1)\pi}{2}$; 30. $\frac{(4k+1)\pi}{2}$ or $\frac{(4k+3)\pi}{2} \pm \frac{\pi}{3}$, 31. $\frac{(2k+1)\pi}{4}$;

32. $\frac{(4k+3)\pi}{4} \leq \theta \leq \frac{(4k+5)\pi}{4}$; 33. $\frac{(6k+1)\pi}{3} \leq \theta \leq \frac{(6k+5)\pi}{3}$;

34. $2k\pi \leq \theta \leq (2k+1)\pi$; 35. $k\pi$; 36. $\frac{(6k+1)\pi}{6} \leq \theta \leq \frac{(6k+5)\pi}{6}$

2. (a) $k\pi$, or $\frac{(4k+1)\pi}{2} \pm \frac{\pi}{3}$; (b) $\frac{(4k+1)\pi}{2}$ or $\frac{(4k+3)\pi}{2} \pm \frac{\pi}{3}$

(c) $(2k+1)\pi \pm \frac{\pi}{3}$; (d) no solution (e) $\frac{(4k+1)\pi}{2}$;

(f) $k\pi \pm \frac{\pi}{3}$; (g) $\frac{(2k+1)\pi}{2}$ or $2k\pi \pm \frac{\pi}{3}$;

(h) $k\pi + \frac{\pi}{4}$; (i) $k\pi \pm \frac{\pi}{3}$; (j) $\frac{(4k-1)\pi}{8}$

MATH ANALYSIS

CONDITIONAL EQUATIONS TRIAL RUN

TRIGONOMETRY

Solve each of the following: (a) aver the domain $0 \le \theta \le 2\pi$;

(b) over all defined domain (Math Analysis Students)

1. $\sin\theta - \tan\theta = 0$
2. $1 - \sin^2\theta = \dfrac{1}{\sec\theta}$
3. $\cos\theta = \dfrac{\cos\theta - 1}{2}$
4. $\sin 2\theta - \sin\theta = 0$
5. $\cos 2\theta = 2 - \cos^2\dfrac{\theta}{2}$
6. $2\cos^2\theta - \sin\theta - 1 = 0$
7. $\cot^2\theta + \csc\theta = 1$
8. $\cos\theta\cot^2\theta - 3\cos\theta = 0$
9. $\dfrac{1 - \cos\theta}{\sin\theta} = \sin\theta$
10. $|\cos\theta| < \dfrac{\sqrt{2}}{2}$
11. $\csc\theta \geq 1$
12. $\cos 2\theta + 6\sin^2\theta = 3$

Answers

(a)

1. $0, \pi, 2\pi$
2. $0, 2\pi$
3. π
4. $0, \pi, 2\pi, \dfrac{\pi}{3}, \dfrac{5\pi}{3}$
5. $0, 2\pi$
6. $\dfrac{\pi}{6}, \dfrac{5\pi}{6}, \dfrac{3\pi}{2}$
7. $\dfrac{7\pi}{6}, \dfrac{11\pi}{6}, \dfrac{\pi}{2}$
8. $\dfrac{\pi}{2}, \dfrac{3\pi}{2}, \dfrac{\pi}{6}, \dfrac{5\pi}{6}, \dfrac{7\pi}{6}, \dfrac{11\pi}{6}$
9. $\dfrac{\pi}{2}, \dfrac{3\pi}{2}$
10. $\dfrac{\pi}{4} < \theta < \dfrac{3\pi}{4}, \quad \dfrac{5\pi}{4} < \theta < \dfrac{7\pi}{4}$
11. $0 < \theta < \pi$
12. $\dfrac{\pi}{4}, \dfrac{3\pi}{4}, \dfrac{5\pi}{4}, \dfrac{7\pi}{4}$

(b)

1. $k\pi$
2. $2k\pi$
3. $(2k + 1)\pi$
4. $k\pi, \quad 2k\pi \pm \dfrac{\pi}{3}$
5. $2k\pi$
6. $\dfrac{(4k + 3)\pi}{2}, \quad \dfrac{(4k + 1)\pi}{2} \pm \dfrac{\pi}{3}$
7. $\dfrac{(4k + 1)\pi}{2}, \quad \dfrac{(4k + 3)\pi}{2} \pm \dfrac{\pi}{3}$
8. $\dfrac{(2k + 1)\pi}{2}, \quad k\pi \pm \dfrac{\pi}{6}$
9. $\dfrac{(2k + 1)\pi}{2}$
10. $\dfrac{(4k + 1)\pi}{4} < \theta < \dfrac{(4k + 3)\pi}{4}$
11. $2k\pi < \theta < (2k + 1)\pi$
12. $k\pi \pm \dfrac{\pi}{4}$