MATH ANALYSIS—ANALYTIC GEOMETRY

LEARNING ACTIVITIES PACKAGE

LIMITS AND CONTINUITY

BEHAVIORAL OBJECTIVES

I. For a given function, $f(x)$, determine $\lim_{x \to a} f(x)$

II. Memorize and use the definition of a limit of a function:

$\lim_{x \to a} f(x) = L$ if and only if $|f(x) - L| < \epsilon$ for $|x - a| < \delta$ for $\epsilon > 0$ and $\delta > 0$.

III. Know the three conditions for a function to be continuous at $x = a$:

A. $f(a)$ exists

B. $\lim_{x \to a} f(x)$ exists and is unique

C. $\lim_{x \to a} f(x) = f(a)$

IV. Use the criteria in III to determine whether or not a function $f(x)$ is continuous at $x = a$.

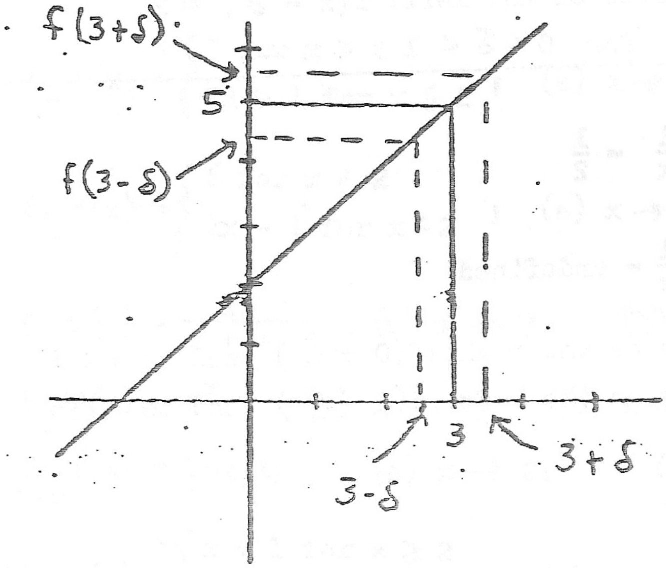
SECTION I. LIMITS--COMMON SENSE APPROACH

Suppose we consider a function $f(x)$ such that $f(x) = x + 2$ and we wonder about the value of $f(x)$ as x gets close to 3. In limit terms, we write $\lim_{x \to 3} (x + 2) = ?$

As x gets close to 3, $(x + 2)$ gets close to 5.

The $\lim_{x \to 3} (x + 2) = 5$. The closer x gets to 3, the closer $f(x)$ gets to 5.

Recall, in graphical terms, $f(x)$ is the y value.



We are not questioning what happens exactly at 3. We ask about the value of the function $f(x)$ for x very close to 3...as close to 3 as we choose to get.

$f(3.1) = 3.1 + 2 = 5.1$

$f(2.9) = 2.9 + 2 = 4.9$

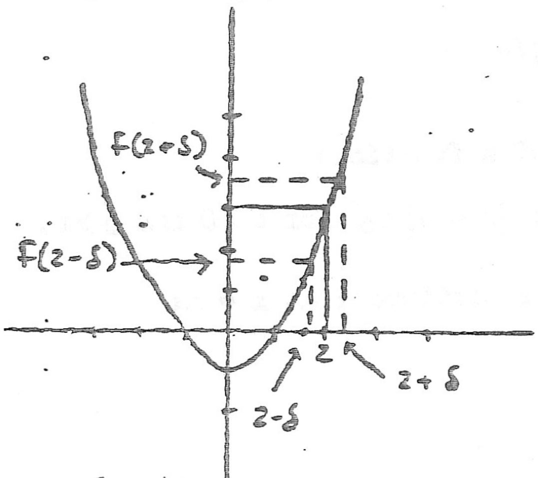
Arbitrarily consider a tiny value δ (delta) such that $\delta > 0$

Then: $f(3 + \delta) = 3 + \delta + 2 = 5 + \delta$

$f(3 - \delta) = 3 - \delta + 2 = 5 - \delta$

Hence, for δ very small, $f(3 \pm \delta)$ is very close to 5.

Example 2: $f(x) = x^2 - 1$



$\lim_{x \to 2} (x^2 - 1) = 3$

$f(2 + \delta) = (2 + \delta)^2 - 1 = 4 + 4\delta + \delta^2 - 1$
$= 3 + 4\delta + \delta^2$

$f(2 - \delta) = (2 - \delta)^2 - 1 = 4 - 4\delta + \delta^2 - 1$
$= 3 - 4\delta + \delta^2$

For $\delta = .1$

$f(2 + \delta) = 3 + .4 + .01 = 3.41$

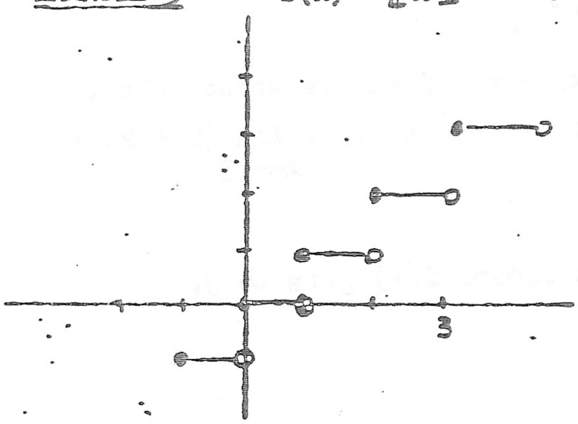
$f(2 - \delta) = 3 - .4 + .01 = 2.61$

For $\delta = .01$:

$f(2 + \delta) = 3 + .04 + .0001 = 3.0401$; $f(2 - \delta) = 3 - .04 + .0001 = 2.9601$

As δ gets smaller, $f(2 \pm \delta)$ gets closer to 3.

EXAMPLE 3 $f(x) = [\![x]\!]$



$\lim_{x \to 3.2} [\![x]\!] = 3$

If we choose deltas very small; i.e. smaller than .2, and close in of 3.2 from both sides, f(x) always equals 3.

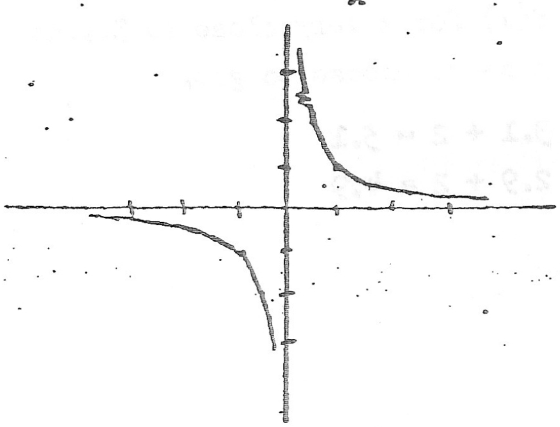
$\lim_{x \to 3} [\![x]\!]$ = not unique

For deltas on the right: $f(x + \delta) = 3$

For deltas on the left: $f(x - \delta) = 2$

for $0 < \delta < 1$

EXAMPLE 4. $f(x) = \frac{1}{x}$



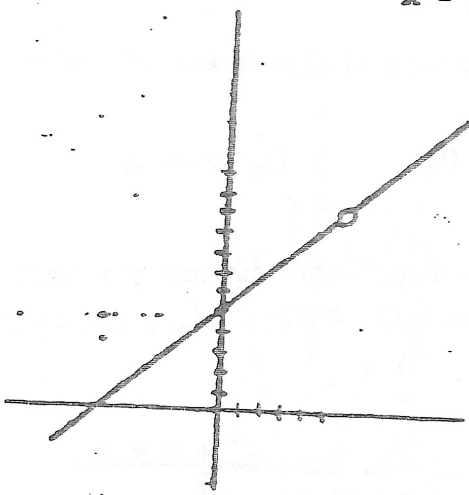
$\lim_{x \to 2} \frac{1}{x} = \frac{1}{2}$

$\lim_{x \to 0} \frac{1}{x}$ = undefined

Deltas on the right: $f(0 + \delta)$ tend to $+\infty$

Deltas on the left: $f(0 - \delta)$ tend to $-\infty$

EXAMPLE 5 $f(x) = \frac{x^2 - 25}{x - 5}$



$\lim_{x \to 5} \frac{x^2 - 25}{x - 5}$

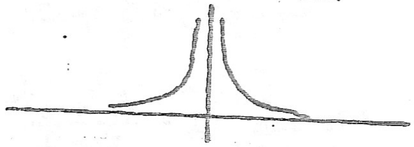
At first glance we might instinctively say the above limit is undefined. To be sure, the function is undefined at $x = 5$. However, in the study of limits we are not concerned with the value of the function at the precise given domain element, but rather values of the function for domain elements close to the given domain element.

$$\lim_{x \to 5} \frac{x^2 - 25}{x - 5} = \lim_{x \to 5} \frac{(x + 5)(x - 5)}{x - 5}$$

$$= \lim_{x \to 5} (x + 5)$$

$$= 10$$

EXAMPLE 6 $f(x) = \frac{1}{|x|}$



$\lim_{x \to 0} \frac{1}{|x|} = \infty$

The limit does not exist. A limit must necessarily be a number.

EXERCISE I

Sketch the graph and determine the limit of the function as x approaches each given domain element.

1. $f(x) = x + 1$; $x \to 5$
2. $f(x) = x^2 - 4$; $x \to 1$
3. $f(x) = \frac{x^2 - 4}{x + 2}$; $x \to -2$
4. $f(x) = [\![x + 1.5]\!]$; (a) $x \to 2$; (b) $x \to .5$
5. $f(x) = \begin{cases} 3 \text{ for } x > 2 \\ 2x - 1 \text{ for } x \leq 2 \end{cases}$; (a) $x \to 3$; (b) $x \to 0$; (c) $x \to 5$
6. $f(x) = \begin{cases} 6 \text{ for } x > 2 \\ 2x - 1 \text{ for } x \leq 2 \end{cases}$; (a) $x \to 3$; (b) $x \to 0$; (c) $x \to 5$
7. $f(x) = \frac{1}{x + 4}$; (a) $x \to 3$; (b) $x \to 0$; (c) $x \to 4$
8. $f(x) = 2^x$; (a) $x \to 0$; (b) $x \to -2$; (c) $x \to 2$
9. $f(x) = \log_2 x$; (a) $x \to 2$; (b) $x \to 0$; (c) $x \to 8$
10. $f(x) = \begin{cases} x + 1 \text{ for } x \geq 2 \\ x - 1 \text{ for } x < 2 \end{cases}$; (a) $x \to 3$; (b) $x \to 2$; (c) $x \to 0$

11. $f(x) = \frac{x^2 - 9}{x - 3}$; (a) $x \to 3$; (b) $x \to -3$; (c) $x \to 0$

12. $f(x) = \frac{x^2 - 6x + 8}{x - 2}$; (a) $x \to 0$; (b) $x \to 2$; (c) $x \to -2$

13. $f(x) = \frac{x^3 - 1}{x - 1}$; (a) $x \to 2$; (b) $x \to 0$; (c) $x \to 1$

V. Evaluate each of the following. Do not graph these functions—unless you want to. In evaluating, you must always avoid cases like 0/0, $\infty/0$; $0/\infty$; and ∞/∞. By hook or crook, get rid of those cases.

1. $\lim\limits_{x \to \infty} \frac{3x - 2}{9x - 7}$
2. $\lim\limits_{x \to \infty} \frac{6x^2 + 2x + 1}{6x^2 - 3x + 4}$
3. $\lim\limits_{x \to \infty} \frac{x^2 + x + 2}{x^3 - 1}$
4. $\lim\limits_{x \to 2} \frac{x - 1}{x^2 - 1}$

*5. $\lim\limits_{x \to 2} \frac{\sqrt{x - 2}}{x^2 - 4}$

*6. $\lim\limits_{x \to 1} \frac{x - 1}{\sqrt{x^2 + 3} - 2}$

*7. $\lim\limits_{x \to \infty} \frac{3^x - 3^{-x}}{3^x + 3^{-x}}$

*8. $\lim\limits_{x \to \infty} \frac{3^{-x} - 3^x}{3^{-x} + 3^x}$

*9. $\lim\limits_{x \to 1} \left(1 + \frac{(x + 1)^2}{x - 1}\right)$

*10. $\lim\limits_{x \to 0} \frac{x^2 + x + 1}{x}$

*11. $\lim\limits_{x \to \pi/2} \sin x$

*12. $\lim\limits_{x \to 0} \frac{1 - (1 + x)^3}{x}$

*13. $\lim\limits_{h \to 0} \frac{f(x + h) - f(x)}{h}$ for: (a) $f(x) = 2x$; (b) $f(x) = x^2$; (c) $f(x) = x^3$; (d) $f(x) = \frac{1}{x}$

---

* Math Analysis only.

YOU MIGHT WANT TO REMEMBER:

$(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$

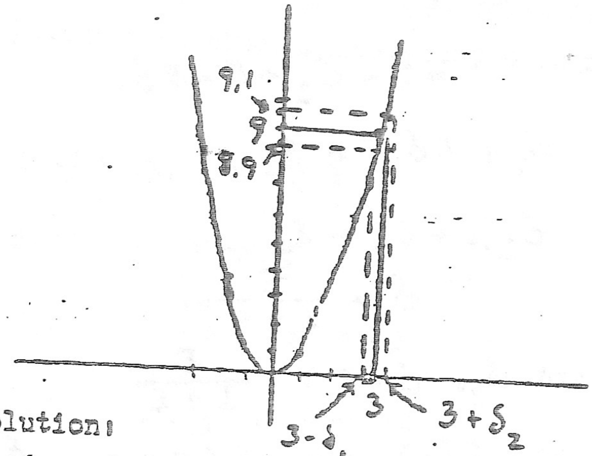
$(a + b)^n = a^n + n\,a^{n-1}b + \frac{n(n-1)a^{n-2}b^2}{2} + \frac{n(n-1)(n-2)a^{n-3}b^3}{3} + \ldots + b^n$

$a^3 + b^3 = (a + b)(a^2 - ab + b^2)$

$a^3 - b^3 = (a - b)(a^2 + ab + b^2)$

SECTION 2 LIMITS—A BIT MORE SOPHISTICATED

Consider the function $f(x) = x^2$



We guess $\lim_{x \to 3} x^2 = 9$

Suppose we want $f(x)$ to be within 0.1 units from 9. We shall later call such a number as 0.1, ε (epsilon.)

Now the question is: How close to 3 must we choose the domain element to insure that this will happen? This choice will be called $\delta(\varepsilon)$; (delta epsilon.)

Solution:

$f(3 + \delta_2) = 9.1$

$(3 + \delta_2)^2 = 9.1$

$3 + \delta_2 = \sqrt{9.1}$

$\delta_2 = \sqrt{9.1} - 3$

$\delta_2 \doteq .016206$

$f(3 - \delta_1) = 8.9$

$(3 - \delta_1)^2 = 8.9$

$3 - \delta_1 = \sqrt{8.9}$

$\delta_1 = 3 - \sqrt{8.9}$

$\delta_1 \doteq .0167133$

The choice for ε (0.1) is the minimum of δ_1 and δ_2.

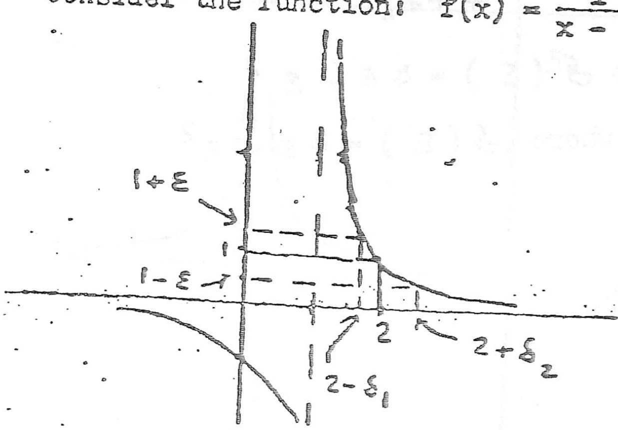
We conclude that: $|f(x) - 9| < 0.1$ as long as $|x - 3| < 0.16206$

Make a great effort to understand the above. Why should we choose the smaller of the two deltas for the final statement?

Numbers, as you can see, can get tedious. From now on instead of chosing a particular number such as 0.1, we shall generalize to any small positive number and call it ε (epsilon.)

EXAMPLE 2

Consider the function: $f(x) = \dfrac{1}{x - 1}$



What about $\lim_{x \to 2} \dfrac{1}{x - 1}$?

1. $\lim_{x \to 2} \dfrac{1}{x - 1} = 1$

2. Determine the relationship between ε and δ such that $|f(x) - 1| < \varepsilon$

Study the diagram. Perhaps draw the diagram to have it close at hand. The solution is on the next ...

Solution:

$$f(2+\delta_2) = 1 - \varepsilon$$
$$\frac{1}{2+\delta_2-1} = 1-\varepsilon$$
$$\frac{1}{\delta_2+1} = 1-\varepsilon$$
$$1 = \delta_2 + 1 - \varepsilon\delta_2 - \varepsilon$$
$$\varepsilon = \delta_2 - \varepsilon\delta_2$$
$$\varepsilon = (1-\varepsilon)\delta_2$$
$$\delta_2 = \frac{\varepsilon}{1-\varepsilon}$$

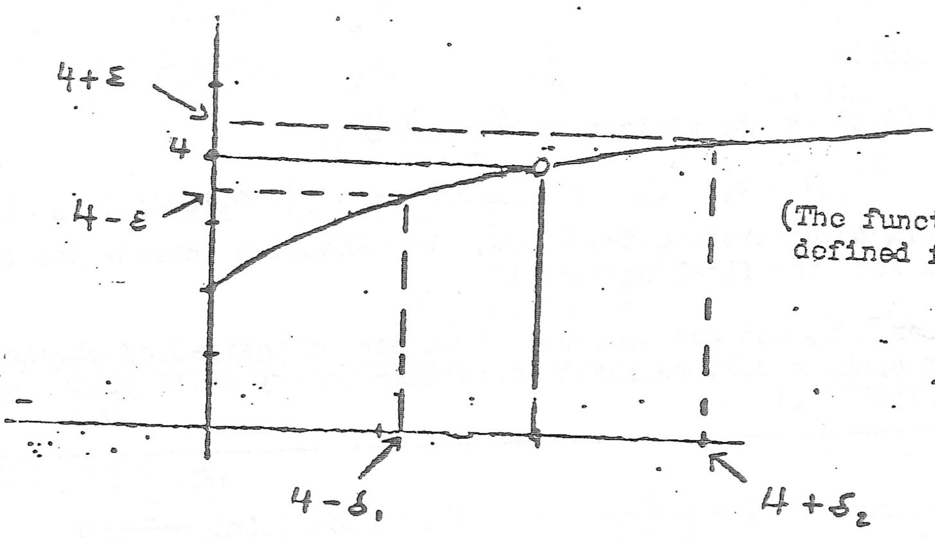
$$f(2-\delta_1) = 1 + \varepsilon$$
$$\frac{1}{2-\delta_1-1} = 1+\varepsilon$$
$$\frac{1}{1-\delta_1} = 1+\varepsilon$$
$$1 = 1 - \delta_1 + \varepsilon - \varepsilon\delta_1$$
$$\delta_1 + \varepsilon\delta_1 = \varepsilon$$
$$\delta_1(1+\varepsilon) = \varepsilon$$
$$\delta_1 = \frac{\varepsilon}{1+\varepsilon}$$

Since $\varepsilon > 0$, δ_1 is the minimum of δ_1 and δ_2. $\delta(\varepsilon) = \dfrac{\varepsilon}{1+\varepsilon}$

Hence: $|f(x) - 1| < \varepsilon$ as long as $|x - 2| < \delta(\varepsilon)$ where $\delta(\varepsilon) = \dfrac{\varepsilon}{1+\varepsilon}$

EXAMPLE 3 $\quad f(x) = \dfrac{x-4}{\sqrt{x}-2}$

$$\lim_{x\to 4} \frac{x-4}{\sqrt{x}-2} = \lim_{x\to 4} \frac{(\sqrt{x}+2)(\sqrt{x}-2)}{\sqrt{x}-2} = \lim_{x\to 4} \sqrt{x}+2 = 4$$



(The function is not defined for x = 4.)

Using the above diagram and algebra, show that $\delta(\varepsilon) = 4\varepsilon - \varepsilon^2$

Hence: $|f(x) - 4| < \varepsilon$ for $|x - 4| < \delta(\varepsilon)$ where $\delta(\varepsilon) = 4\varepsilon - \varepsilon^2$

EXERCISE 2

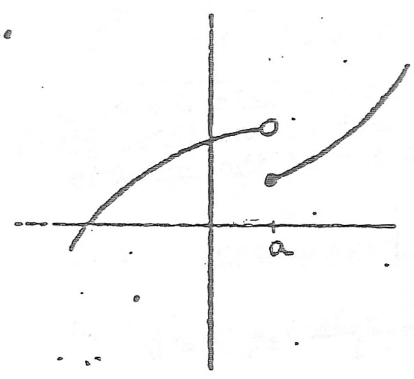
For each of the following: (a) Graph; (b) Guess the limit as $x \to a$ where a is the number indicated in each problem; (c) for $f(x + \delta_2)$, solve for δ_2 in terms of ε; (d) For $f(x - \delta_1)$, solve for δ_1 in terms of ε; (e) Determine $\delta(\varepsilon)$; (f) Make a concluding statement such as $|f(x) - L| < \varepsilon$ for $|x - a| < \delta(\varepsilon)$ where $\delta(\varepsilon) =$ _____

1. $f(x) = 3x; \quad x \to 1$
2. $f(x) = 2x - 3; \quad x \to 4$
3. $f(x) = \frac{2}{x} \quad x \to \frac{1}{2}$
4. $f(x) = -\sqrt{x}; \quad x \to 4$
5. $f(x) = \sqrt{1 - x}; \quad x \to -3$
6. $f(x) = \sqrt[3]{x}; \quad x \to -1$
7. $f(x) = \frac{x^2 - 1}{x + 1}; \quad x \to -1$
8. $f(x) = |x + 2| \quad x \to -4$
9. $f(x) = \frac{\sqrt{x} - 2}{x - 4}; \quad x \to 4$
10. $f(x) = \begin{cases} x \text{ for } x < 0 \\ x^2 \text{ for } x \geq 0 \end{cases}; \quad x \to 0$

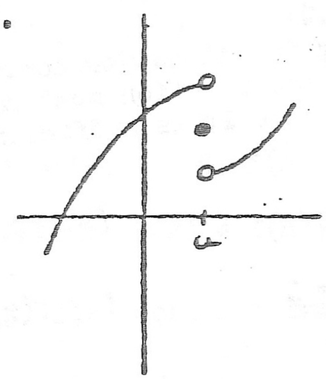
SECTION 3 CONTINUITY

Each of the graphs of functions/relations graphed below illustrate discontinuity for $x = a$. Take a good look.

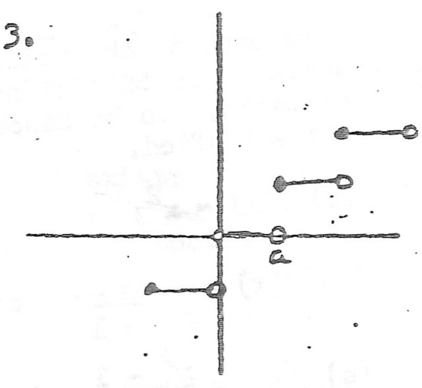
1.



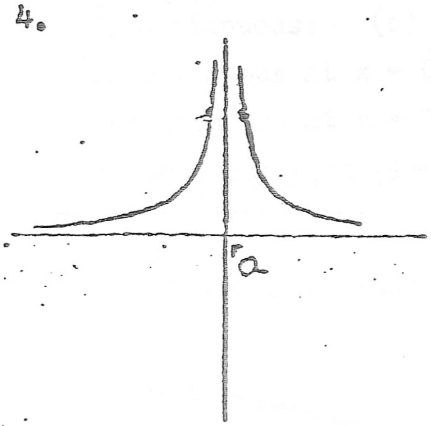
2.



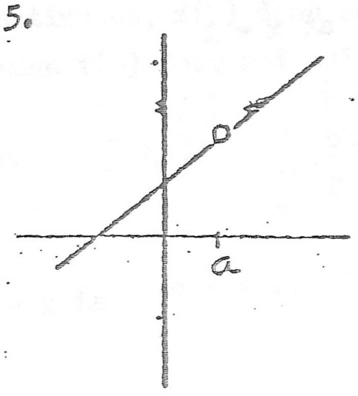
3.



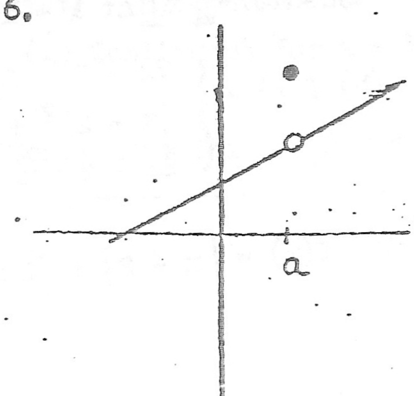
4.



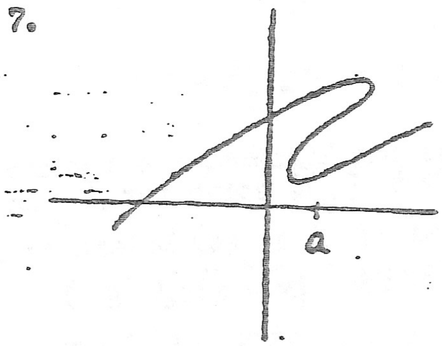
5.



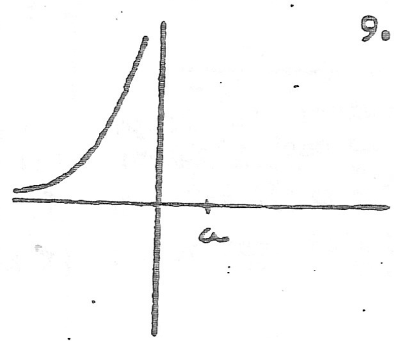
6.



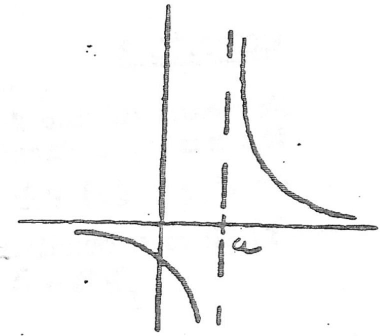
7.



8.



9.



In order for a function to be continuous at a domain element "a", the following three criteria must be satisfied:

1. $f(a)$ exists
2. $\lim_{x \to a} f(x)$ exists and is unique
3. $\lim_{x \to a} f(x) = f(a)$

We include the unique in #2 to cover cases such as $\lim_{x \to 3} [\![x]\!]$. If we approach 3 from the left, $\lim_{x \to 3^-} [\![x]\!] = 2$. If we approach 3 from the right, $\lim_{x \to 3^+} [\![x]\!] = 3$. The greatest integer function is not continuous for $x = 3$. In fact, the greatest integer function is discontinuous for x any integer.

EXERCISE 3

1. Tell why each function graphed at the beginning of this section is not continuous for $x = a$.

2. Test each of the following functions for continuity at x equals the given number. To be continuous the function must satisfy each of the three given criteria. To be discontinuous it is sufficient that one of the criteria is not satisfied.

(a) $f(x) = x^2$; at $x = 0$

(b) $f(x) = |x|$ at $x = -4$

(c) $f(x) = \dfrac{1}{\sqrt{x - 1}}$ at $x = 1$

(d) $f(x) = \dfrac{x^2 - 1}{x - 1}$ at $x = 0$

(e) $f(x) = \dfrac{x^2 - 1}{x - 1}$ at $x = 1$

(f) $f(x) = \dfrac{1}{x}$ at $x = 2$; at $x = 0$

(g) $f(x) = [\![x]\!]$ at $x = 1.5$; at $x = 1$

(h) $f(x) = \begin{cases} 2x - 3, & \text{for } x \leq 2 \\ (x - 2)^2 + 1; & \text{for } 2 < x \leq 3 \\ 2x - 4 & \text{for } x > 3 \end{cases}$ at $x = 2$; at $x = 3$

(i) $f(x) = |x - 2| + |2x + 1|$ at $x = 2$; at $x = -\frac{1}{2}$

ANSWERS

EXERCISE 1

1. 6; 2. −3; 3. −4; 4. (a) 3; (b) not unique, N.E. 5. (a) 3; (b) −1;
5. (c) 3. 6. (a) 6; (b) −1; (c) 6; 7. (a) $\frac{1}{7}$; (b) $\frac{1}{4}$; (c) N.E.
8. (a) 1 (b) $\frac{1}{4}$; (c) 4; 9. (a) 1; (b) $-\infty$ or N.E.; (c) 3; 10. (a) 4;
10. (b) not unique; N.E. (c) −1; 11. (a) 6; (b) 0; (c) 3; 12. (a) −4;
12. ~~(b) −2~~; (c) −6; 13. (a) 7; (b) ~~N.E.~~ (c) 3.

II.
1. $\frac{1}{3}$; 2. 1; 3. 0; 4. $\frac{1}{3}$; 5. N.E.; 6. 2; 7. 1; 8. −1;
9. N.E.; 10. N.E.; 11. 1; 12. −3; 13. (a) 2; (b) $2x$; (c) $3x^2$; (d) $-\frac{1}{x^2}$

EXERCISE 2

1. $|f(x) - 3| < \varepsilon$ for $|x - 1| < \delta(\varepsilon)$, where $\delta(\varepsilon) = \frac{\varepsilon}{3}$
2. $\delta(\varepsilon) = \frac{\varepsilon}{2}$; 3. $\delta(\varepsilon) = \frac{\varepsilon}{8 + 2\varepsilon}$; 4. $\delta(\varepsilon) = 4\varepsilon - \varepsilon^2$;
5. $\delta(\varepsilon) = 4\varepsilon - \varepsilon^2$; 6. $\delta(\varepsilon) = 3\varepsilon - 3\varepsilon^2 + \varepsilon^3$; 7. $\delta(\varepsilon) = \varepsilon$
8. $\delta(\varepsilon) = \varepsilon$; 9. $\delta(\varepsilon) = 4 - \left(\frac{2 - 8\varepsilon}{1 + 4\varepsilon}\right)^2$; 10. $\delta(\varepsilon) = \varepsilon$.

EXERCISE 3

1. 1, 2, and 3, $\lim_{x \to a} f(x)$ is not unique. 4. f(a) does not exist;
5. f(a) does not exist; 6. $\lim_{x \to a} f(x) \neq f(a)$; 7. not a function
8. f(a) does not exist. 9. f(a) does not exist.

2. (a) Continuous; (b) Continuous; (c) Discontinuous, f(1) does not exist;
(d) Continuous; (e) Discontinuous, f(1) does not exist; (f) Continuous at 2;
Discontinuous at x = 0 because f(0) does not exist; (g) Continuous for x = 1.5
~~Discontinuous at $x = 1$ because $\lim_{x \to 1} f(x)$ is not unique~~
(h) Continuous at x = 2; Continuous at x = 3. (i) Continuous for x=2 & x = $-\frac{1}{2}$.

N.E. DOES NOT EXIST

MATH ANALYSIS—ANALYTIC GEOMETRY LIMITS—TRIAL RUN

I. Determine the limit for each of the following: (*Math Analysis only.)

1. $\lim_{x \to \infty} \frac{1}{x^2 + 3}$

2. $\lim_{x \to \infty} \frac{x^2}{1 - x^3}$

3. $\lim_{x \to -1} \frac{x}{x - 3}$

4. $\lim_{x \to 0} \frac{1}{x}$

5. $\lim_{x \to -1} (2x^2 + 5x - 2)$

6. $\lim_{x \to 0} \frac{3^x - 3^{-x}}{3^x + 3^{-x}}$

7. $\lim_{x \to 4} \frac{x + 4}{x^2 - 16}$

8. $\lim_{x \to 3} \frac{x^3 - 27}{x^2 - 9}$

9. $\lim_{x \to 2} \frac{4 - x^2}{3 - \sqrt{x^2 + 5}}$

10. $\lim_{x \to 1} \frac{x^2 + x - 2}{(x - 1)^2}$

11. $\lim_{x \to -1} \frac{x^3 + 1}{x + 1}$

*12. $\lim_{x \to -2} \frac{x^3 - 2x^2 - 5x + 6}{x^2 + x - 2}$

*13. $\lim_{x \to \pi} (x + \sin x)$

*14. $\lim_{x \to \pi} (x + \cos x)$

*15. $\lim_{x \to .4} [\![2x + 1]\!]$

II. For each of the following: 1. Graph; 2. Guess $\lim_{x \to a} f(x)$; 3. Determine δ_2 in terms of ε; 4. Determine δ_1 in terms of ε. 5. Determine $\delta(\varepsilon)$; 6. Complete the statement $|f(x) - L| < \varepsilon$ for $|x - a| < \delta(\varepsilon)$.

1. $f(x) = x - 7$; $x \to 2$;

2. $f(x) = \sqrt{2 - x}$; $x \to -2$

3. $f(x) = |2x - 1|$; $x \to 2$

4. $f(x) = \frac{3}{x}$; $x \to \frac{1}{3}$

5. $f(x) = \frac{4 - x^2}{x + 2}$; $x \to -2$

*6. $f(x) = \log_3 x$; $x \to 3$

III. Write the three criteria for a function $f(x)$ to be continuous at $x = a$.

IV. Determine whether or not each of the functions below are continuous for the specified domain element. Justify your answer.

1. $f(x) = [\![x + .7]\!]$; at $x = .3$

2. $f(x) = x^2 + 2x$; for $x = 5$

3. $f(x) = \frac{\sqrt{x} - 4}{x - 16}$; for $x = 16$

4. $f(x) = x^3 + 2$; for $x = 2$

5. $f(x) = [\![x + .7]\!]$; for $x = 1$

6. $f(x) = \sqrt{x^2 - 4}$; for $x = 0$

7. $f(x) = \begin{cases} 2x & \text{for } x > 2 \\ 0 & \text{for } x \leq 2 \end{cases}$ for $x = 2$ 8. $f(x) = \begin{cases} \frac{1}{x} & \text{for } x \neq 0 \\ 0 & \text{for } x = 0 \end{cases}$ for $x = 0$

9. $f(x) = \begin{cases} \frac{x^2 - 1}{x - 1} & \text{for } x \neq 1 \\ 2 & \text{for } x = 1 \end{cases}$; $x=1$ 10. $f(x) = \begin{cases} \frac{x^2 - 1}{x - 1} & \text{for } x \neq 1 \\ 1 & \text{for } x = 1 \end{cases}$; for $x = 1$

ANSWERS:

I. 1. 0; 2. 0; 3. $\frac{1}{4}$; 4. N.E, undefined; 5. -5; 6. 0; 7. N.E, undefined; 8. $\frac{9}{2}$; 9. 6; 10. N.E, undefined; 11. 3; 12. -5; 13. π; 14. $\pi - 1$; 15. 1.

II. 1. $|f(x) + 5| < \varepsilon$ for $|x - 2| < \delta(\varepsilon)$; $\delta(\varepsilon) = \varepsilon$

2. $|f(x) - 2| < \varepsilon$ for $|x + 2| < \delta(\varepsilon)$; $\delta(\varepsilon) = 4\varepsilon - \varepsilon^2$

3. $|f(x) - 3| < \varepsilon$ for $|x - 2| < \delta(\varepsilon)$; $\delta(\varepsilon) = \frac{\varepsilon}{2}$

4. $|f(x) - 9| < \varepsilon$ for $|x - \frac{1}{3}| < \delta(\varepsilon)$; $\delta(\varepsilon) = \frac{\varepsilon}{27 + 3\varepsilon}$

5. $|f(x) - 4| < \varepsilon$ for $|x + 2| < \delta(\varepsilon)$; $\delta(\varepsilon) = \varepsilon$

6. $|f(x) - 1| < \varepsilon$ for $|x - 3| < \delta(\varepsilon)$; $\delta(\varepsilon) = 3(1 - 3^{-\varepsilon})$

IV. 2.; 4.; 5.; and 9 are continuous. All criteria are satisfied.

1. Not continuous because $\lim_{x \to .3} f(x)$ is not unique.

3. Not continuous because f(16) does not exist.

6. Not continuous because f(0) does not exist.

7. Not continuous because $\lim_{x \to 2} f(x)$ is not unique.

8. Not continuous because $\lim_{x \to 0} f(x)$ does not exist.

10. Not continuous because $\lim_{x \to 1} f(x) \neq f(1)$.