MATH ANALYSIS-ANALYTIC GEOMETRY

PARABOLA
LEARNING ACTIVITIES PACKAGE

BEHAVIORAL OBJECTIVES

I. Define

A. A parabola as a locus of points
B. A parabola as a conic section

II. For a given parabola identify its

A. Vertex
B. Focus
C. Axis of symmetry
D. Directrix
E. Latus rectum

III. Given the equation of a parabola determine

A. The coordinates of its
1. Vertex
2. Focus

B. The equations of its
1. Axis of symmetry
2. Directrix

C. The length of the latus rectum

D. Focal length

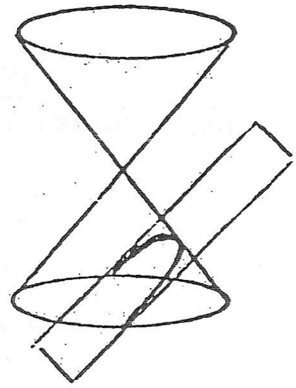
IV. Given the equation of a parabola in standard form transfer the equation to the informational form

V. Determine the equation of a parabola given

A. The focus and the directrix
B. Three points on the parabola
C. The focus and the vertex
D. The endpoints of the latus rectum and the directrix

SECTION I PARABOLA--DEFINITION

The parabola is the intersection of a plane with a right circular double napped cone such that the plane is parallel to an edge of the cone.



The parabola: Locus definition: A parabola is the locus of all points in a plane whose distances from a fixed point equal their distances from a fixed line.

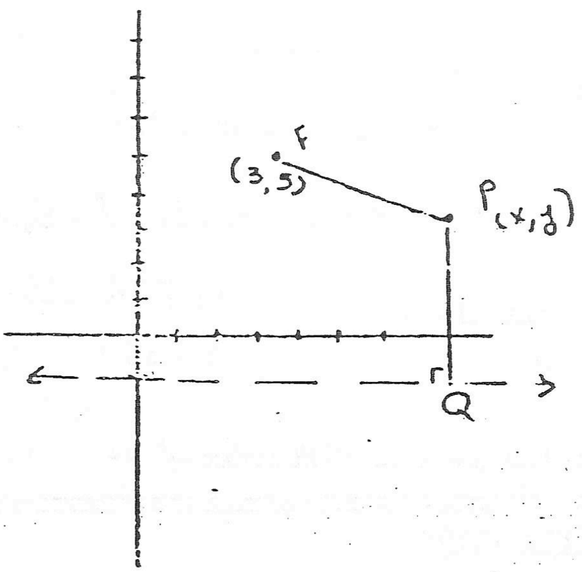
The fixed point in the above definition is called the focus.

The fixed line in the above definition is called the directrix.

NOTE: As in the ellipse, the ratio of the distance from a point on the conic to the the focus to the distance of that point on the conic to the directrix, defines the eccentricity of the conic. In an ellipse the eccentricity was a number between zero and one. For the parabola, the eccentricity is always one, since the distance to the focus is equal to the distance to the directrix.

By using the above definition we can equation of a parabola if we are given the focus and the directrix. Study the examples that follow:

EXAMPLE 1 Given a focus at (3,5) and a directrix $y = -1$, write the equation of the parabola determined and sketch the graph of the parabola.



Choose an arbitrary point P on the conic with coordinates (x,y). By difinition of the parabola, PF = PQ.

Hence:

$$\sqrt{(x - 3)^2 + (y - 5)^2} = |y + 1| \quad (1)$$

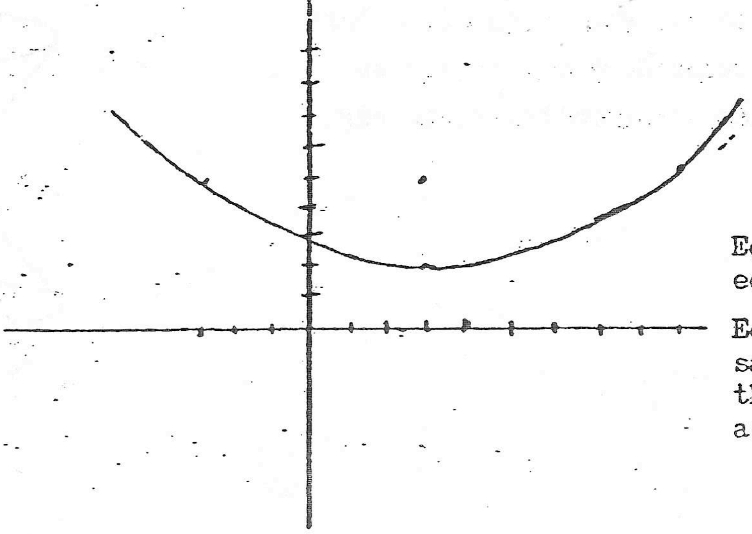
For the above equation, square both sides and simplify to show that the equation is:

$$(x - 3)^2 = 12(y - 2) \quad (2)$$

Notice that the point (3,2) is midway between the focus and the directrix.

(3,2) is the vertex of the parabola.

Find several other ordered pairs that fit the equation to sketch the curve.



Equation (1) above is the equation of the parabola.

Equation (2) is really the same equation but in a form that will be very helpful as we procede.

EXERCISE 1:

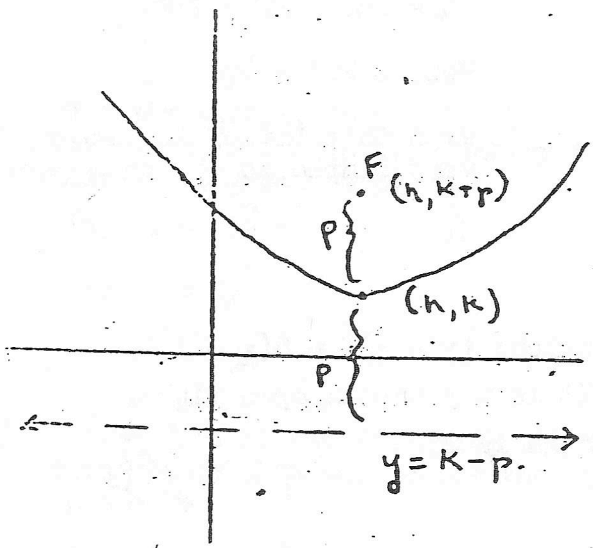
For each of the following find the equation of the parabola defined. Write the equation in the form: $(x - h)^2 = a(y - k)$ OR $(y - k)^2 = a(x - h)$. Also GRAPH.

1. Focus: (3,4) and directrix $x = 1$
2. Focus (3,4) and directrix $x = 5$
3. Focus (3,4) and directrix $y = 2$
4. Focus: (3,4) and directrix $y = 6$.
5. Focus (4,0) and directrix $x = -4$
6. Focus: (0,4) and directrix $y = -4$
7. Focus (4,0) and directrix $x = 8$
8. Focus (0,4) and directrix $y = 8$.
9. Focus (0,0) and directrix $x = 10$
10. Focus (0,0) and directrix $y = 7$

SECTION II THE ANATOMY OF A PARABOLA

In at least one person's opinion, the parabola is the most beautiful of all the conics. Possibly this is so because of its simplicity. With an eccentricity of 1, all parabolas are similar. To obtain the general informational equation for the parabola we can procede as follows:

Suppose the vertex of the parabola is (h,k). Let the distance from the vertex to the focus be p. Then the coordinates of the focus are (h,k+p) for F above the vertex and $p > 0$. The directrix is $y = k - p$.



By definition of the parabola we can set up this equation:

$$\sqrt{(x - h)^2 + (y - (k+p))^2} = |y - (k - p)|$$

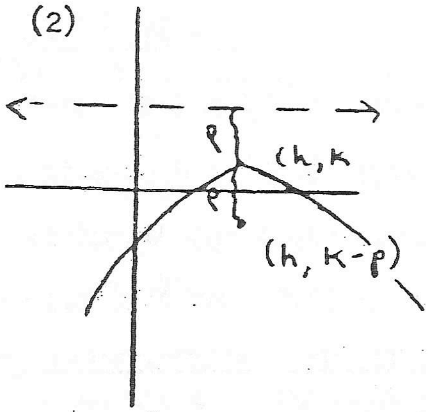
Square both sides, combine like terms and show that the above equation simplifies to:

$$(x - h)^2 = 4p(y - k)$$

The vertex is (h,k)
The focal distance is p.
The axis of symmetry of the parabola is $x = h$
The focus is (h,k+p)
The directrix is $y = k - p$

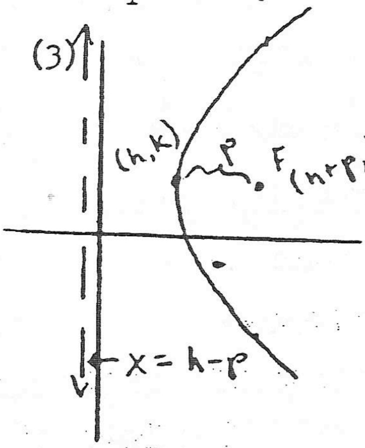
On the previous page we considered one arrangement of the vertex and the focus. In this L.A.P. we shall also meet three other placements of the vertex and the focus. With each the procedure used before can be employed to derive the information form of the desired equation. For reference purposes, we shall show below the remaining three positions and their equations:

(2)



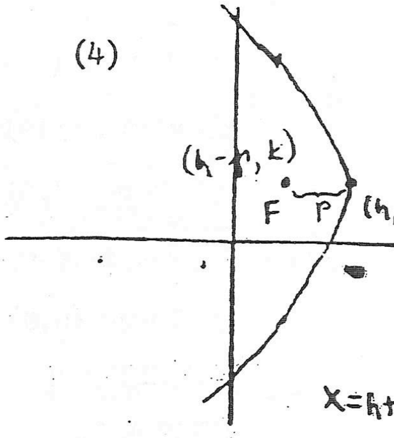
$(x - h)^2 = -4p(y - k)$

(3)



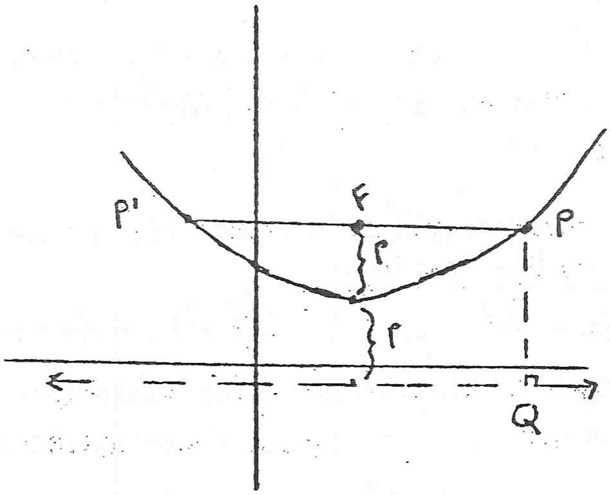
$(y - k)^2 = 4p(x - h)$

(4)



$(y - k)^2 = -4p(x - h)$

The Latus rectum is absolutely nifty! Look:



The segment PP' is the latus rectum.

PP' contains the focus and is paralle to the directrix.

PF = PQ by the definition of the coni

PQ = 2p. PF = P'F.

Hence: PP' = 4p.

With this bit of knowledge, it is very simple to sketch a parabola.

EXAMPLE: Sketch the parabola with equation: $(y - 3)^2 = 8(x - 1)$.

The equation is of type (3) above. It is a parabola open right.

The vertex is (1,3) and the focal length is 2.

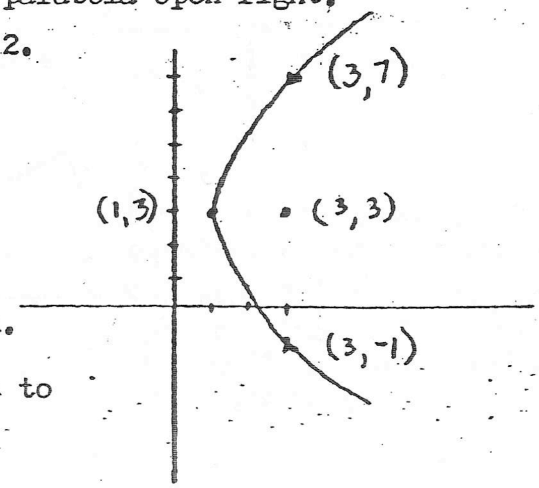
Plot (1,3).

Locate the focus 2 units to the right.

Then go up 4 units from the focus to locate a point on the parabola.

Then go down 4 units from the focus to locate another point on the parabola.

The three points thus located are sufficient to sketch the curve.

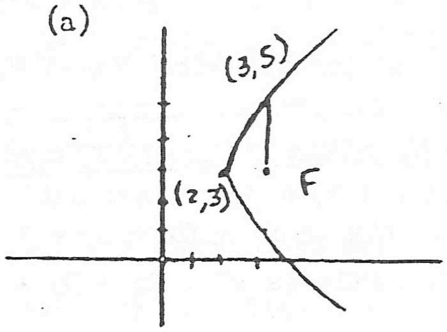


ALWAYS PLOT THE VERTEX AND END POINTS OF THE LATUS RECTUM WHEN SKETCHING A PARABOLA!

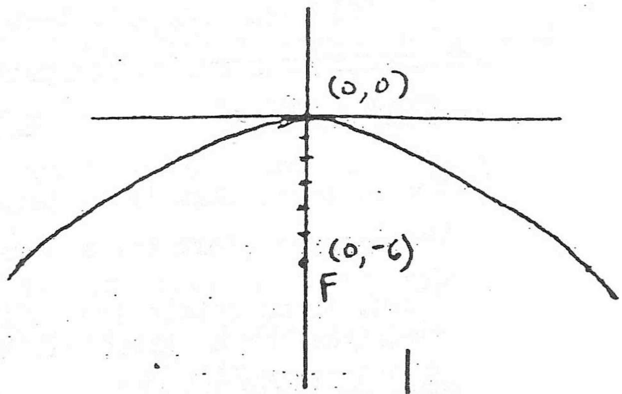
EXERCISE 2

1. Write the equation of each parabola sketched below:

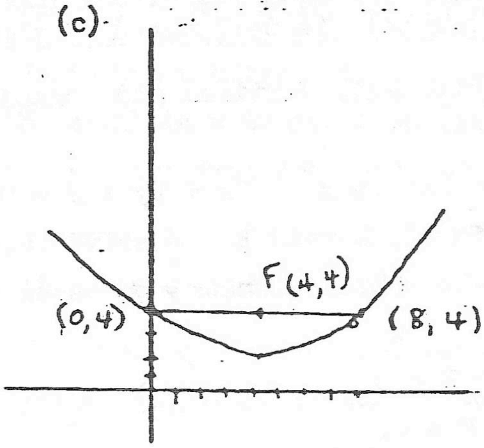
(a)



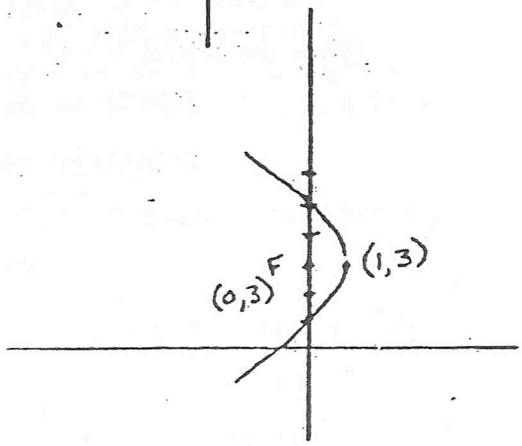
(b)



(c)



(d)



2. Find the vertex, focus, focal length, length of latus rectum, equation of directrix, and axis of symmetry for each of the following. Sketching the graph will be of great help.

(a) $(x - 2)^2 = -12(y - 3)$ (b) $(x + 1)^2 = 4(y + 1)$

(c) $x^2 = 8y$ (d) $(y + 2)^2 = 20x$

(e) $y^2 = -5x$ (f) $(y - 3)^2 = -10(x - 3)$

3. Change each of the following equations of parabolas to the informational form. (Here we go with completing the square again!) Identify the vertex of the parabola.

(a) $x^2 + 2x - y = 0$ (b) $2x^2 + 4x - 3y = 0$

(c) $2x^2 + 4x + 5y = 0$ (d) $5y^2 - 10y + 3x = 0$

(e) $x^2 - 2x - 4y + 5 = 0$ (f) $y^2 + 2y + 7x = 0$

4. Write the equation of each parabola described below:

(a) Vertex at (2,8) and focus at (2,4)

(b) Focus at (1,1) and directrix $y = 7$.

(c) Endpoints of the latus rectum (8,4) and (0,4), parabola open down.

(d) Equation of the directrix $x = 2$, and vertex $(0,0)$

(e) Focus $(2,-6)$, axis of symmetry $x = 2$, and focal length 2. (two solutions)

(f) Focus $(-2,-1)$, latus rectum endpoints: $(-2,2)$ and $(-2,-4)$. (Two solutions)

SECTION III THREE POINTS! YES, SIR, COULD DETERMINE A PARABOLA!

The standard form for a parabola with vertical axis is $x^2 + Cx + Dy + E = 0$.
The standard form for a parabola with horizontal axis is $y^2 + Cx + Dy + E = 0$.

Hence, three points are sufficient to determine the equation of a parabola with a vertical or a horizontal axis. NOTE: No parabola is determined if the three points are collinear.

EXAMPLE: Find the eqution of the parabola with axis vertical and contains the points $(4,5)$, $(-2,11)$, and $(-4,21)$.

Solution: The standard equation to use is $x^2 + Cx + Dy + E = 0$.

We need to find values for C, D, and E, to solve the problem.

Set up three equations with three unknowns by using the given.

$$16 + 4C + 5D + E = 0$$
$$4 - 2C + 11D + E = 0$$
$$16 - 4C + 2D + E = 0.$$

Solve the above system of equations to show $C = -4$, $D = -2$ and $E = 10$.

Hence, the desired equation is $x^2 - 4x - 2y + 10 = 0$.

EXERCISE 3

Write the equation of each parabola determined:

1. Contains the points $(3,3)$, $(6,5)$, and $(6,-3)$, and has a horizontal axis.
2. Contains the points $(-2,1)$, $(1,2)$, $(-1,3)$ and has a horizontal axis.
3. Contains the points $(5,0)$, $(1,2)$ and $(3,1)$. Vertical axis.
4. Contains the points $(5,0)$, $(1,2)$ and $(3,4)$. Two solutions.

MATH ANALYSIS STUDENTS GO ON TO SECTION IV.

ANALYTIC GEOMETRY STUDENTS (1) Review the L.A.P.; (2) Take the Trial Run; (3) Take the test.

SECTION IV GETTING SOPHISTICATED

No new concepts are dealt with in thei section. We simply take what we know and make extended use in a variety of situations. Review the concepts from the previous section and try to be creative. Do not worry about a little brain pain!

EXERCISE 4

1. Find the equation of the line containing the points on the parabola $y^2 = 8x$ whose y coordinates are 2 and 8 respectively.

2. Write the equation of the circle which contains the vertex of the parabola $(x - 2)^2 = 4(y + 1)$, and is tangent to the latus rectum of the parabola.

3. Write the equation of the ellipse whose major axis is the latus rectum of the parabola with equation $(y + 1)^2 = 8(x + 1)$ and whose semi-minor axis is the seqment from the focus to the vertex of the parabola.

4. A parabolic arch has a height of 25 feet and a span of 40 feet. How high is the arch 8 feet each side of the center of the span?

5. Find the equation of the parabola whose axis is parallel to the Y-axis with vertex (1,3) and contains the point (5,7).

6. Find the equation of the parabola with vertex at the origin, contains the point (2,3) and has its axis along the X-axis.

7. Write the equation of each parabola described below. These parabolas have axes which are not parallel to the coordinate axes and therefore do not have such "nice" equations. Use the the definition of a parabola; i.e. the locus of points which are equidistant from a fixed point and a fixed line, to create your solutions.

   (a) Directrix $x = y$ and focus (3,10).

   (b) Directrix $y = -2x$ and focus (4,4).

   (c) Directrix $3x + 4y - 2 = 0$ and focus (0,0).

8. Show that the circle with the latus rectum of the parabola $x^2 = 4py$ as a diameter is tangent to the directrix.

9. Prove that the lines from the endpoints of the latus rectum of a parabola to the point of intersection of the axis of symmetry and the directrix of the parabola are perpendicular to each other.

SECTION V. EVALUATION

1. Review the L.A.P. 2. Take the Trial Run. 3. Take the test.

ANSWERS

EXERCISE 1

1. $(y - 4)^2 = 4(x - 2)$; 2. $(y - 4)^2 = -4(x - 4)$
3. $(x - 3)^2 = 4(y - 3)$ 4. $(x - 3)^2 = -4(y - 5)$ 5. $y^2 = 16x$ 6. $x^2 = 1$
7. $y^2 = -8(x - 6)$ 8. $x^2 = -8(y - 6)$ 9. $y^2 = -20(x - 5)$
10. $x^2 = -14(y - \frac{7}{2})$

EXERCISE 2

1. (a) $(y - 3)^2 = 4(x - 2)$ (b) $x^2 = -24y$ (c) $(x - 4)^2 = 8(y - 2)$
   (d) $(y - 3)^2 = -4(x - 1)$
2. (a) (2,3); (2,0), 3; 12; $y = 6$; $x = 2$. (b) (-1,-1); (-1,0); 1; 4; $Y = -2$;x
   (c) (0,0); (0,2); 2; 8; $y = -2$; $x = 0$. (d) (0,-2); (5,-2); 5; 20; $x = -5$; y =
   (e) (0,0); $(-\frac{5}{4},0)$; $\frac{5}{4}$; 5; $x = \frac{5}{4}$; $y = 0$. (f) (3,3); $(\frac{1}{2},3)$; $\frac{5}{2}$; 10; $x = \frac{11}{2}$; y = 3
3. (a) $(x + 1)^2 = y + 1$; (-1,-1). (b) $(x + 1)^2 = \frac{3}{2}(y + \frac{2}{3})$; $(-1, -\frac{2}{3})$
   (c) $(x - 1)^2 = -\frac{5}{2}(y - \frac{2}{5})$; $(1,\frac{2}{5})$. (d) $(y - 1)^2 = -\frac{3}{5}(x - \frac{5}{3})$; $(\frac{5}{3}; 1)$
   (e) $(x - 1)^2 = 4(y - 1)$; (1,1) (f) $(y + 1)^2 = -7(x - \frac{1}{7})$; $(\frac{1}{7},-1)$
4. (a) $(x - 2)^2 = -16(y - 8)$ (b) $(x - 1)^2 = -12(y - 4)$
   (c) $(x - 4)^2 = -8(y - 6)$ (d) $y^2 = -8x$ (e) $(x - 2)^2 = -8(y + 4)$ and
   (e) $(x - 2)^2 = 8(y + 8)$; (f) $(y + 1)^2 = -6(x + \frac{1}{2})$ and $(y + 1)^2 = 6(x + \frac{7}{2})$

EXERCISE 3

1. $y^2 - 4x - 2y + 9 = 0$; 2. $5y^2 + 2x - 21y + 20 = 0$; 3. No parabola...collinea
4. $3x^2 - 16x + 4y + 5 = 0$ and $3y^2 - 4x - 14y + 20 = 0$

EXERCISE 4

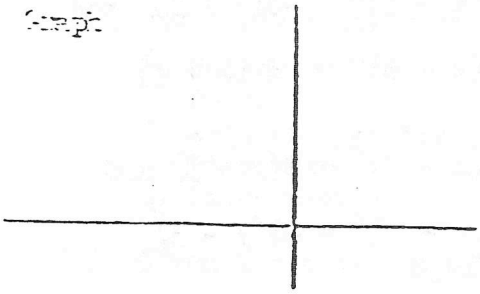
1. $5y = 4x + 8$ 2. $(x - 2)^2 + (y + \frac{1}{2})^2 = \frac{1}{4}$
3. $\frac{(x - 1)^2}{4} + \frac{(y + 1)^2}{16} = 1$ 4. 21 feet; 5. $(x - 1)^2 = 4(y - 3)$
6. $y^2 = \frac{9}{2}x$ 7. (a) $(x - 3)^2 + (y - 10)^2 = \frac{(x - y)^2}{2}$
   (b) $(x - 4)^2 + (y - 4)^2 = \frac{(2x + y)^2}{5}$ (c) $x^2 + y^2 = \frac{(3x + 4y - 2)^2}{25}$

PARABOLA TRIAL RUN

I. Define: A. Parabola; B. Focus of a parabola; C. Latus rectum of a parabola D. Axis of symmetry of a parabola

II. Given: $(y - 2)^2 = 12(x + 3)$

Graph



A. Coordinates of the vertex: ________

B. Coordinates of the focus: ________

C. Equation of the directrix: ________

D. Equation of the axis: ________

E. Length of latus rectum: ________

III. Write the equation of each parabola described below:

1. Focus (3,4) and directrix $x = 7$
2. Vertex (0,4) and latus rectum of length 18, Vertical axis. (Two solutions.)
3. Vertex (2,4), axis of symmetry $y = 4$, contains the point (-10,6)
4. Endpoints of the latus rectum (2,-3) and (10,-3). (Two solutions)
5. Vertex (10,5) and focus (10,0).
6. Vertex (5,3), focal length 2, vertical axis. (Two solutions.)

IV. Change each of the following standard form equations to the informational form. For each parabola identify the vertex and the focus.

1. $y^2 - 4y + 6x - 8 = 0$
2. $3x^2 - 9x - 5y - 2 = 0$
3. $y^2 - 4y - 6x + 13 = 0$
4. $8x^2 + 16x - 12y + 5 = 0$

V. Write the equation of the parabola which

1. Contains the points (3,1), (2,0), (1,1) and has a vertical axis.
2. Contains the points (3,3), (2,0), (1,1) and has a horizontal axis.
3. Contains the points (3,3), (2,0), (1,1) and has a vertical axis.

VI. Show by a diagram how a plane must cut a right circular double napped cone so that the intersection will be a parabola.

VII. What is the value of the eccentricity of a parabola?

A FEW FOR MATH ANALYSIS EXERCISE

1. Find the equation of the line which contains the foci of the parabolas $y^2 = 4x$ and $x^2 = -8y$.
2. Find the equation of the circle which has its center at the origin and contains a point of intersection of the graphs of $y^2 = 4x$ and $x^2 = 4y$.
3. Approximate the measure of the angle formed by segments from the endpoints of the latus rectum of a parabola to the vertex of the parabola.
4. Review the last section of the L.A.P.

PARABOLA TRIAL RUN ANSWERS

II. A. (-3,2); B. (0,2); C. $x = -6$; D. $y = 2$; E. 12

III. 1. $(y - 4)^2 = -8(x - 5)$

2. $x^2 = 18(y - 4)$ and $x^2 = -18(y - 4)$

3. $(y - 4)^2 = -\frac{1}{3}(x - 2)$

4. $(x - 6)^2 = -8(y + 1)$ and $(x - 6)^2 = 8(y + 5)$

5. $(x - 10)^2 = -20(y - 5)$

6. $(x - 5)^2 = 8(y - 3)$ and $(x - 5)^2 = -8(y - 3)$

IV. 1. $(y - 2)^2 = -6(x - 2)$

2. $(x - \frac{3}{2})^2 = \frac{5}{3}(y + \frac{7}{4})$

3. $(y - 2)^2 = 6(x - \frac{3}{2})$

4. $(x + 1)^2 = \frac{3}{2}(y + \frac{1}{4})$

V. 1. $x^2 - 4x - y + 4 = 0$

2. $2y^2 - 3x - 5y + 6 = 0$

3. $2x^2 - 7x - y + 6 = 0$

VII. 1

MATH ANALYSIS

1. $y = 2x - 2$

2. $x^2 + y^2 = 32$

3. 2 Arc tan 2. This is approximately 127°