MATH ANALYSIS-ANALYTIC GEOMETRY

SEQUENCES AND SERIES

LEARNING ACTIVITIES PACKAGE

BEHAVIORAL OBJECTIVES

I. Distinguish between

- A. A sequence and a series
- B. A convergent and a divergent sequence
- C. A convergent and a divergent series

II. Give an example of

- A. An arithmetic series
- B. A geometric series
  - 1. Which converges
  - 2. Which diverges
- C. A series which converges by oscillation

III Given the formula for the general term of a sequence write

- A. Any specified term
- B. The first five terms of the sequence
- C. Guess whether the sequence
  - 1. Converges
  - 2. Diverges
- D. If the sequence converges, guess the limit of the sequence

IV. Given a series in sigma notation

- A. Write the first five terms of the series
- *B. Determine its sequence of partial sums
- *C. Guess whether the series is
  - 1. Convergent
  - 2. Divergent
- *D. Employ the comparison test to establish convergence or divergence of a series

V. Find the sum of a given arithmetic series to a specified number of terms

VI. Find the sum of a given geometric series

- A. To a given number of terms
- B. As the number of terms tends to infinity wher $|r| < 1$.

VII. Given a series write the series using sigma notation.

VIII. Write a given number of terms of

- *A. A p-series which
  - 1. Converges
  - 2. Diverges
- B. The Fibonacci sequence
- *C. The harmonic sequence

IX. Evaluate the limit of a given expression in n as n tends to infinity

*X. Derive

A. The formula for the sum of an arithmetic series
B. The formula for the sum of a geometric series
C. The formula for the limit of the sum of a geometric series as n tends to infinity where $|r| < 1$.

*XI. Use the Principle of Mathematical Induction to prove theorems about Series.

* Math Analysis Students only.

SECTION I SEQUENCES

A sequence is a function whose domain is the set of positive integers. If the domain is the entire set of positive integers the sequence is infinite. If the domain is a finite subset of the positive integers, the sequence is finite.

Example: $a(1) = \frac{1}{2}$; $a(2) = \frac{2}{3}$; $a(3) = \frac{3}{4}$; . . . $a(n) = \frac{n}{n+1}$ . . .

For convenience we write: $a_1 = \frac{1}{2}$; $a_2 = \frac{2}{3}$; $a_3 = \frac{3}{4}$; . . . $a_n = \frac{n}{n+1}$; . . .

Or, even shorter: $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \ldots \frac{n}{n+1}, \ldots$

In an arithmetic sequence a common difference exists between each two successive terms.

Example: 1, 3, 5, 7, 9, . . . 2n-1, . . . (The common difference is 2.)

In a geometric sequence a common ratio exists between each two successive terms.

Example: 1, 3, 9, 27, . . . , 3^{n-1}, . . . (The common ratio is 3.)

Then, we can make up all kinds of crazy sequences. A sequence does not have to be either arithmetic or geometric. Some have no predictable behavior at all. Some have a defining rule. These latter, at times, allow us to make some interesting observations of patterns.

Look at the following examples of sequences. Study the side notes.

1. 4, -2, 3, $\sqrt{21}$, 49, 71, ... (Hopeless! There is absolutely no pattern to this sequence.)

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2. 1, 2, 6, 24, 120, . . . (It appears that this sequence might be the factorial sequence. It is not fair to put such a label on it. The next term might well be -1. However, here we will. Under the condition that the sequence continues as it is we can generalize to say $a_n = n!$.)

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3. 1, 1, 2, 3, 5, 8, 13, ...

(This is a classic. It is called the Fibonacci Sequence. Each term, after the first two, is obtained by adding the two terms preceding it. The next term would be 21. In general, $a_n = a_{n-1} + a_{n-2}$ for $n > 2$.

As n gets very large, a_n gets very large. Such a sequence is said to <u>diverge</u>.)

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4. 1, -1, 1, -1, 1, ...

(The general rule for this dull sequence is

$$a_n = (-1)^{n-1}$$

No unique number is reached as n gets very large. This sequence is also said to <u>diverge</u>.)

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5. $\frac{1}{2}, -\frac{1}{6}, \frac{1}{12}, -\frac{1}{20}, \ldots$

(Very interesting! With good magic glasses and a little luck we see that the general term $a_n = (-1)^{n-1} \frac{1}{n^2 + n}$

The sequence oscillates. As n gets larger and larger, a_n gets closer and closer to 0. One time larger than 0, and the next time smaller than 0, but ever getting closer to 0. Such a sequence is said to <u>converge</u>. 0 is the limit of the sequence as n tends to infinity.)

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6. $\frac{1}{3}, \frac{2}{5}, \frac{3}{7}, \frac{4}{9}, \ldots$

(Not bad. Would you believe that $a_n = \frac{n}{2n + 1}$. For better insight we divide through by n to write $a_n = \frac{1}{2 + \frac{1}{n}}$

In this latter form it is easy to see that a_n thends to $\frac{1}{2}$ as n gets very large, because $1/n$ tends to 0 as n gets very large. This sequence <u>converges</u>. The limit of the sequence as n tends to infinity is $\frac{1}{2}$.

<u>NOTE</u>: If the limit of an infinite sequence is a real number, the sequence is said to <u>converge</u>. If an infinite sequence does not converge it is said to <u>diverge</u>.

Study the section carefully and then go on to the exercises.

EXERCISES #1

1. Write the first 3 terms of each of the following sequences:

(a) $a_n = 1 + \frac{1}{n}$ (b) $a_n = \frac{1}{n(n+1)}$ (c) $a_n = \frac{n}{\sqrt{1+n^2}}$

(d) $a_n = \frac{\sqrt{n+1}}{n}$ (e) $a_n = (-1)^{n-1}\frac{n!}{n^n}$ (f) $a_n = 3n + 1$

(g) $a_n = n(2^n)$ (h) $a_n = (\frac{1}{2})^n$

2. For each sequence: (a) Write two more terms; (b) Guess a formula for a_n;
(c) As n gets very large, what can you say about a_n?
(d) Does the sequence converge or diverg?

A. $\frac{3}{2}, \frac{6}{3}, \frac{9}{4}, \frac{12}{5}, \ldots$ B. 1, 4, 9, 16, . . .

C. 2,4, 6, 8, . . . D. 0, 7, 26, 63, . . .

E. $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$ F. $\frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{1}{32}, \ldots$

G. $\frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}, \ldots$ H. 3, 5, 7, 9, 11, ...

I. $\frac{1}{2}, \frac{2}{5}, \frac{3}{8}, \frac{4}{11}, \ldots$ J. $1, \sqrt{2}, \sqrt{3}, 2, \sqrt{5}, \ldots$

K. 1, 0, 1, 0, 1, 0, . . . L. $1, -\frac{1}{2}, \frac{1}{3}, -\frac{1}{4}, \frac{1}{5}, \ldots$

M. -1, 4, -1, 8, -1, 12, . . . N. 0, log 4, log 7, 1, log 13, . . .

3. Which of the sequences in #1 are:
(a) Arithmetic? (b) Geometric? (c) Convergent? (d) Divergent?
(e) Monotonic increasing? (Always getting bigger as n gets bigger.)
(f) Monotonic decreasing?

4. Repeat #3 for the sequences in #2.

5. Write the first 15 terms of the Fibonacci Sequence.

Rather that always saying what about a_n as n gets very large, the "big guys" say $\lim_{n \to \infty} a_n =$ _____. This is read, the limit of a_n as n tends to infinity.

Guess the $\lim_{n \to \infty} a_n$ for each of the following. (Hint: divide through by the highest power of n before attempting to evaluate.) [Test your guess by evaluating some large values of n.]

A. $a_n = \frac{1}{n}$

B. $a_n = \frac{1}{n^2}$

C. $a_n = \frac{1}{n^3}$

D. $a_n = \frac{n}{n + 3}$

E. $a_n = \frac{3n}{n + 5}$

F. $a_n = \frac{n^2 + 2n + 1}{3n^2 + n}$

G. $a_n = \frac{5n^3 + 4n^2}{6n^3 + n}$

H. $a_n = \frac{n^2 + 2n + 1}{n^2 + n - 1}$

I. $a_n = \frac{6n}{n^2 + 1}$

J. $a_n = \frac{8}{n^3}\left[\frac{(n - 1)(2n - 1)n}{6}\right]$

K. $a_n = \log \frac{2n}{n - 1}$

L. $a_n = n^{\frac{1}{n}}$

M. $a_n = \frac{\sin n\pi + 4n}{n - 1}$

N. $a_n = \frac{\cos n\pi + 4n}{n + 1}$

SECTION II ARITHMETIC AND GEOMETRIC SERIES

A series is the expressed sum of a sequence. So, for the sequence: 2, 4, 6, 8,... 2n, the corresponding series is 2 + 4 + 6 + 8 + . . . + 2n. Using sigma notation this series can be written: $\sum_{j=1}^{n} 2j$.

A finite series is the expressed sum of a finite sequence.

Example: 2, 4, 6, 8, 10. The series is: 2 + 4 + 6 + 8 + 10.

In sigma notation it is: $\sum_{n=1}^{5} 2n$.

An infinite series is the expressed sum of an infinite series. For the above it would be: 2 + 4 + 6 + 8 + . . .

Using sigma notation we write: $\sum_{n=1}^{\infty} 2n$

If a sequence is arithmetic, so is its corresponding series. If a sequence is geometric, so is its corresponding series.

The two old faithfuls, arithmetic and geometric, series have handy formulas to use to determine the sums.

ARITHMETIC: Example: $2 + 4 + 6 + 8$

$8 + 6 + 4 + 2$ (Write it backwards.)

$10 + 10 + 10 + 10$ (Add, term by term.)

So, the sum is: $\frac{4 \times 10}{2}$. We divide by 2 because we have really used the series twice.

In general, an arithmetic series looks like this:

$$S_n = [a_1] + [(a_1 + d)] + [(a_1 + 2d)] + [(a_1 + 3d)] + \ldots + [(a_1 + (n-1)d)]$$

OR $$S_n = [a_1 + (n-1)d] + [(a_1 + (n-2)d)] + [(a_1 + (n-3)d)] + \ldots + [a_1]$$

Add: $$2S_n = (2a_1 + (n-1)d) + (2a_1 + (n-1)d) + (2a_1 + (n-1)d) + \ldots + (2a_1 + (n-1)d)$$

So: $$2S_n = n(2a_1 + (n-1)d)$$

Hence: $$\boxed{S_n = \frac{n(2a_1 + (n-1)d)}{2}}$$

In an arithmetic sequence, $a_n = a_1 + (n-1)d$

With the above substitution, we can generate an alternate formula for the sum of an arithmetic series. It is:

$$\boxed{S_n = \frac{n(a_1 + a_n)}{2}}$$

Both of the above formulas are worth remembering. Math Analysis students be able to derive them. An infinite arithmetic series cannot be summed. As $n \to \infty$, so does S_n.

For a geometric series:

$$S_n = a_1 + a_1(r) + a_1(r^2) + a_1(r^3) + a_1(r^4) + \ldots + a_1(r^{n-1})$$

Then: $$rS_n = a_1(r) + a_1(r^2) + a_1(r^3) + a_1(r^4) + \ldots + a_1(r^{n-1}) + a_1(r^n)$$

Subtract: $S_n - rS_n = a_1 - a_1(r^n)$

$$S_n(1 - r) = a_1(1 - r^n)$$

$$\boxed{S_n = \frac{a_1(1 - r^n)}{1 - r}}$$

Notice, when $|r| < 1$, $\lim_{n \to \infty} r^n = 0$. Hence, if a geometric series has a ratio "r" such that $|r| < 1$, the series has a sum, even if we consider an infinite number of terms!

$$\boxed{\text{For } |r| < 1, \quad S_\infty = \frac{a_1}{1 - r}}$$

It seems incomprehensible that adding an infinite number of terms leads to a finite sum!

> " 'I can't believe that,' said Alice.
>
> " 'Can't you?' the Queen said in a pitying tone. 'Try again; draw a long breath and shut your eyes.'
>
> "Alice laughed: 'There's no use trying,' she said; 'one can't believe impossible things.'
>
> "' I daresay you haven't had much practice,' said the Queen. 'When I was younger, I always did it for half an hour a day. Why, sometimes I've believed as many as six impossible things before breakfast!' " THROUGH THE LOOKING GLASS, Carroll

Actually, we can't properly speak of the sum of an infinite series because we could never complete the job of adding the terms. The sum of an infinite series is the limit of the sum of the series of n terms as n tends to infinity.

For instance: $1 + \frac{1}{2} + \frac{1}{4} + \ldots + (\frac{1}{2})^{n-1} + \ldots$ never gets beyond 2. The sum of this infinite geometric series is 2. Such a series is said to converge.

<u>MATH ANALYSIS STUDENTS</u>: Be able to derive both formulas discussed on this page.

EXERCISE 2

1. The following series are either arithmetic or geometric. Continue each through 6 terms. Write the resulting series using sigma notation.

A. $8 + 4 + 2 + \ldots$

B. $3 + 6 + 9 + \ldots$

C. $\frac{1}{4} + \frac{1}{16} + \frac{1}{64} + \ldots$

D. $\frac{1}{3} - 1 + 3 - \ldots$

E. $a^2 + a^4 + a^6 + \ldots$

F. $i - 1 - i + \ldots$

G. $4 + 7 + 10 + \ldots$

H. $22 + 17 + 12 + \ldots$

2. If c is the arithmetic mean between a and b then a, c, b form an arithmetic sequence.

If c is the geometric mean between a and b then a, c, b, form a geometric sequence.

Find the arithmetic mean and the geometric mean for each pair of numbers given below:

A. 16 and 20

B. $\sqrt{2}$ and 9

C. -4 and -8

D. 1 and 1

E. -4 and 8

3. A. Prove that the arithmetic mean between a and b is $\frac{a + b}{2}$.

B. Prove that the geometric mean between a and b is $\sqrt{ab}$.

4. Find the sum of each of the following series

A. $\sum_{n=1}^{12} (2n + 1)$

B. $\sum_{n=1}^{4} 3 \cdot (2)^{n-1}$

C. $\sum_{n=1}^{8} (6n - 5)$

D. $\sum_{n=1}^{\infty} 4(\frac{1}{2})^{n-1}$

E. $\sum_{n=1}^{35} (4n - 6)$

F. $\sum_{n=1}^{12} 6 (\frac{1}{4})^n$

G. $\sum_{n=1}^{k} (2n - 1)$

H. $\sum_{n=1}^{k} n$

I. $\sum_{n=1}^{k} 3(8)^{n-1}$

5. How many terms of the sum of $1 + 3 + 5 + \ldots + (2n-1) + \ldots$ are needed to give a sum of 12321?

6. What distance will a golf ball travel if it is dropped from a height of 72", and if, after each fall it rebounds .9 of the distance it fell?

7. Find the sum of the even integers from 10 to 58 inclusive.

8. An equilateral triangle has a perimeter of 12 cm. By joining the midpoints of its sides with line segments a new triangle is formed. Suppose this is continued for each new triangle formed. Find the sum of the perimeters of all triangles including the original one.

9. A square has a perimeter of 40 cm. By joining the midpoints of its sides with line segments a new square is formed. This is continued for each new square formed. Find the sum of the perimeters of all squares including the original one. Find the sum of the areas of all squares including the original one.

10. Circles A, B, C, ... are internally tangent at point P. The diameter of circle A is 100 cm., and each circle has respectively, a diameter equal in length to one-half the diameter of the previous circle. Find the sum of the circumferences of all the circles.

11. Prove each of the following using the Principle of Mathematical Induction:

A. $\sum_{j=1}^{n}(2j - 1) = n^2$ B. $\sum_{j=1}^{n}(3j - 2) = \frac{n(3n - 1)}{2}$

C. $\sum_{j=1}^{n} j^3 = \frac{n^2(n + 1)^2}{4}$ D. $\sum_{j=1}^{n}(j \cdot j!) = (n + 1)! - 1$

SECTION III TESTS FOR CONVERGENCE AND DIVERGENCE OF SERIES

MATH ANALYSIS ONLY.

A necessary, but not sufficient, condition for the convergence of a series is that the $\lim_{n \to \infty} a_n = 0$. As n gets very large the terms must get closer and closer to zero. If $\lim_{n \to \infty} a_n \neq 0$, then the series diverges. If $\lim_{n \to \infty} a_n = 0$, we cannot conclude that the series converges! Sorry about that!

The comparison test is often employed to establish convergence or divergence of a series.

1. If a given series is term by term smaller than a known series which converges, the given series converges.

2. If a given series is term by term larger than a known series which diverges, the given series diverges.

One can also study a given series' sequence of partial sums. The $\lim_{n \to \infty}$ of the sequence of partial sums is the sum of the series. If the sequence of partial sums diverges, so does the series. If the limit of the sequence of partial sums is some finite number, that number is the sum of the series. Such a series converges.

STUDY THE FOLLOWING EXAMPLES:

1. $S = 1 + 2 + 3 + 4 + \ldots + n + \ldots$

$a_n \to \infty$ as $n \to \infty$. Since it is necessary for convergence that the nth term tend to zero as n tends to infinity, we conclude that this series diverges.

2. $S = 1 + \frac{1}{2} + \frac{1}{4} + \ldots + (\frac{1}{2})^{n-1} + \ldots$

This series is geometric with $r = \frac{1}{2}$. The series has a sum of 2. Hence, this series is said to converge.

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3. $S = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots + \frac{1}{n} + \ldots$

The above series is called the <u>Harmonic Series</u>. The harmonic series diverges. The divergence of the harmonic series is not immediately apparent. Proof is given below. The harmonic series is often used in the comparison test to establish divergence of a given series.

$$S = 1 + \frac{1}{2} + (\frac{1}{3} + \frac{1}{4}) + (\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}) + (\frac{1}{9} + \ldots + \frac{1}{16}) + \ldots$$

$$S = 1 + \frac{1}{2} + \frac{7}{12} + (\text{a number} > \tfrac{1}{2}) + (\text{a number} > \tfrac{1}{2}) + \ldots$$

$$S > 1 + \tfrac{1}{2} + \tfrac{1}{2} + \tfrac{1}{2} + \ldots$$

Consequently, the series diverges, because it is greater than another series which diverges. Why does $1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \ldots$ diverge?

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4. $S = 1 + \frac{1}{2^p} + \frac{1}{3^p} + \frac{1}{4^p} + \ldots + \frac{1}{n^p} + \ldots$ for $p > 1$

$$S = 1 + (\frac{1}{2^p} + \frac{1}{3^p}) + (\frac{1}{4^p} + \frac{1}{5^p} + \frac{1}{6^p} + \frac{1}{7^p}) + (\frac{1}{8^p} + \ldots + \frac{1}{15^p}) + \ldots$$

$$S < 1 + (\frac{1}{2^p} + \frac{1}{2^p}) + (\frac{1}{4^p} + \frac{1}{4^p} + \frac{1}{4^p} + \frac{1}{4^p}) + (\frac{1}{8^p} + \ldots + \frac{1}{8^p}) + \ldots$$

$$S < 1 + \frac{2}{2^p} + \frac{4}{4^p} + \frac{8}{8^p} + \ldots$$

$$S < 1 + \frac{1}{2^{p-1}} + \frac{1}{4^{p-1}} + \frac{1}{8^{p-1}} + \ldots$$

Hence, S is smaller than the geometric series with $a_1 = 1$ and $r = \frac{1}{2^{p-1}}$.

For $p > 1$, $|r| < 1$, hence the geometric series converges and the above series which is smaller must then also converge.

Any series of the form: $1 + \frac{1}{2^p} + \frac{1}{3^p} + \ldots + \frac{1}{n^p}$, is called a <u>p-series.</u>

<u>The p-series converges for $p > 1$.</u> <u>The p-series diverges for $p \leq 1$.</u>

Discuss with your teacher and/or other friends why the p-series diverges for $p \leq 1$.

5. $S = \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \ldots + \frac{1}{n(n+1)} + \ldots$

This series converges. It is not directly obvious that this is so. The nth term tends to zero as n tends to infinity. Hence the necessary condition is satisfied. However, that condition is not sufficient. Consider the sequence of partial sums:

$S_1 = \frac{1}{2}$

$S_2 = \frac{1}{2} + \frac{1}{6} = \frac{4}{6} = \frac{2}{3}$

$S_3 = S_2 + \frac{1}{12} = \frac{2}{3} + \frac{1}{12} = \frac{9}{12} = \frac{3}{4}$

$S_4 = S_3 + \frac{1}{20} = \frac{3}{4} + \frac{1}{20} = \frac{16}{20} = \frac{4}{5}$

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$S_n = \frac{n}{n+1}$ This is a generalization. It can be proved using the Principle of Mathematical Induction.

$\lim_{n \to \infty} \frac{n}{n+1} = 1.$ Hence, the sum of the series is 1. The series converges to 1.

EXERCISE 3

1. Each of the following series diverg because: (1) $\lim_{n \to \infty} a_n \neq 0$; (2) It is term by term greater than the harmonic series; (3) It is geometric with $|r| \geq 1$; or (4) It is a p-series with $p \leq 1$.

Give a valid reason for the divergence of each of the following series:

A. $1 + \frac{1}{2^{-1}} + \frac{1}{3^{-1}} + \ldots + \frac{1}{n^{-1}} + \ldots$

B. $\frac{1}{3} + \frac{1}{2} + \frac{3}{4} + \ldots + \frac{1}{3}(\frac{3}{2})^{n-1} + \ldots$

C. $6 + 12 + 24 + \ldots + 5(2)^{n-1} + \ldots$

D. $2 + 1 + \frac{2}{3} + \ldots + \frac{2}{n} + \ldots$

E. $1 + \frac{5}{8} + \frac{10}{18} + \ldots + \frac{n^2+1}{2n^2} + \ldots$

F. $\frac{1}{.9} + \frac{1}{1.9} + \frac{1}{2.9} + \ldots + \frac{1}{n - .1} + \ldots$

2. Identify each of the following series as convergent or divergent. Justify your answer.

A. $1. + \frac{1}{2} + \frac{1}{4} + \ldots + (\frac{1}{2})^{n-1} + \ldots$ B. $\frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \ldots + (\frac{1}{2})^{n+1} + \ldots$

C. $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots + \frac{1}{n} + \ldots$ D. $1 + \frac{1}{2^3} + \frac{1}{3^3} + \ldots + \frac{1}{n^3} + \ldots$

E. $\frac{2}{3} + \frac{4}{9} + \frac{8}{27} + \ldots + (\frac{2}{3})^n + \ldots$ F. $2 + 4 + 8 + 16 + \ldots + 2n + \ldots$

G. $\frac{1}{2} + \frac{1}{8} + \frac{1}{32} + \ldots + (\frac{1}{2})^{2n-1} + \ldots$ H. $1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \ldots + \frac{1}{\sqrt{n}} + \ldots$

3. Each of the following is a geometric series which converges. Find the number to which it converges.

A. $4 + 2 + 1 + \ldots + 4(\frac{1}{2})^{n-1} + \ldots$ B. $\frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \ldots + (\frac{1}{3})^n + \ldots$

C. $.7 + .07 + .007 + \ldots + 7(.1)^n + \ldots$ D. $2 + \frac{2}{3} + \frac{2}{9} + \ldots + 2(\frac{1}{3})^{n-1} + \ldots$

E. $47 + 47(.01) + \ldots + 47(.01)^{n-1} + \ldots$ F. $3 - 1 + \frac{1}{3} - \frac{1}{9} + \ldots$

4. For each of the following determine: (a) S_1, S_2, S_3, and S_4; (b) S_n; (c) $\lim_{n \to \infty} S_n$; (d) whether the series converges or diverges; (e) the number to which it converges if the series is convergent.

A. $1 + \frac{1}{3} + \frac{1}{6} + \ldots + \frac{2}{n^2 + n} + \ldots$ B. $1 + \frac{1}{5} + \frac{3}{35} + \ldots + \frac{3}{4n^2 - 1} + \ldots$

C. $\frac{5}{2} + \frac{5}{6} + \frac{5}{12} + \ldots + \frac{5}{n^2 + n} + \ldots$ D. $\frac{1}{2} + \frac{5}{6} + \frac{11}{12} + \ldots + \frac{n^2 + n - 1}{n^2 + n} + \ldots$

E. $1 + \frac{1}{4} + \frac{5}{44} + \ldots + \frac{10}{9n^2 + 3n - 2} + \ldots$ F. $\frac{3}{2} + \frac{1}{2} + \frac{1}{4} + \ldots + \frac{3}{n^2 + n} + \ldots$

G. $\frac{1}{1 \cdot 3} + \frac{1}{3 \cdot 5} + \ldots + \frac{1}{(2n - 1)(2n + 1)}$

SECTION IV. EVALUATION

1. Review the Behavioral Objectives.

2. Take the Trial Run.

3. Take the test.

ANSWERS

EXERCISE 1

1. (a) 2, $\frac{3}{2}$, $\frac{4}{3}$ (b) $\frac{1}{2}$, $\frac{1}{6}$, $\frac{1}{12}$ (c) $\frac{1}{\sqrt{2}}$, $\frac{2}{\sqrt{5}}$, $\frac{3}{\sqrt{10}}$ (d) $\sqrt{2}$, $\frac{\sqrt{3}}{2}$, $\frac{2}{3}$

   (e) 1, $-\frac{1}{2}$, $\frac{2}{9}$ (f) 4, 7, 10 (g) 2, 8, 24 (h) $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{8}$

2.

	(a)	(b)	(c)	(d)
A.	$\frac{15}{6}$, $\frac{18}{7}$	$\frac{3n}{n+1}$	tends to 3	Converges
B.	25, 36	n^2	gets large	Diverges
C.	10, 12	2n	gets large	Diverges
D.	124, 215	$n^3 - 1$	gets large	Diverges
E.	$\frac{1}{5}$, $\frac{1}{6}$	$\frac{1}{n}$	tends to 0	Converges
F.	$\frac{1}{64}$, $\frac{1}{128}$	$(\frac{1}{2})^{n+1}$	tends to 0	Converges
G.	$\frac{6}{7}$, $\frac{7}{8}$	$\frac{n+1}{n+2}$	tends to 1	Converges
H.	13, 15	2n + 1	gets large	Diverges
I.	$\frac{5}{14}$, $\frac{6}{17}$	$\frac{n}{3n-1}$	tends to $\frac{1}{3}$	Converges
J.	$\sqrt{6}$, $\sqrt{7}$	$\sqrt{n}$	gets large	Diverges
K.	1, 0	$\frac{1}{2} + \frac{1}{2}(-1)^{n-1}$	oscillates	Diverges
L.	$-\frac{1}{6}$, $\frac{1}{7}$	$(\frac{1}{n})(-1)^{n-1}$	tends to 0	converges
M.	-1, 16	-1 for n odd 2n for n even	oscillates	Diverges
N.	log 16, log 19	log (3n - 2)	gets large	Diverges

3. (a) f; (b) h; (c) a, b, c, d, e, h; (d) f, g; (e) c, f, g;

   (f) a, b, d, h.

4. (a) C, H; (b) F; (c) A, E, F, G, I, L; (d) B, C, D, H, J, K, M, N;

   (e) A, B, C, D, G, H, J, N; (f) E, F, I.

5. 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, . . .

6. A. 0; B. 0; C. 0; D. 1; E. 3; F. $\frac{1}{3}$; G. $\frac{5}{6}$; H. 1

   I. 0; J. $\frac{8}{3}$; K. log 2; L. 1; M. 4; N. 4.

EXERCISE 2

1. $1 + \frac{1}{2} + \frac{1}{4}$ B. $12 + 15 + 18$ C. $\frac{1}{4^4} + \frac{1}{4^5} + \frac{1}{4^6}$

D. $-9 + 27 - 81$ E. $a^8 + a^{10} + a^{12}$ F. $1 + i - 1$

G. $13 + 16 + 19$ H. $7 + 2 - 3$

1. Sigma notation:

A. $\sum_{n=1}^{6} 8(\frac{1}{2})^{n-1}$ B. $\sum_{n=1}^{6} 3n$ C. $\sum_{n=1}^{6} (\frac{1}{4})^n$ D. $\sum_{n=1}^{6} (-1)^{n-1}(3)^{n-2}$

E. $\sum_{n=1}^{6} a^{2n}$ F. $\sum_{n=1}^{6} i^n$ G. $\sum_{n=1}^{6} (3n + 1)$ H. $\sum_{n=1}^{6} (-5n + 27)$

2. Arithmetic mean: A. 18; B. $\frac{9 + \sqrt{2}}{2}$; C. -6; D. 1; E. 2

Geometric mean: A. $8\sqrt{5}$; B. $3\sqrt[4]{2}$; C. $-4\sqrt{2}$; D. 1; E. No G.M.

3. A. Let d = the common difference, and x be the arithmetic mean.

Then, a, x, b is an arithmetic sequence. $b = a + 2d$; $x = a + d$.

From $b = a + 2d$ it follows that $d = \frac{b - a}{2}$. Hence: $x = a + \frac{b-a}{2} = \frac{a + b}{2}$.

B. Let r = the common ratio, and x be the geometric mean.

Then, a, x, b is a geometric sequence. $b = a \cdot r^2$; $x = a \cdot r$.

From $b = a \cdot r^2$ it follows that $r = \sqrt{\frac{b}{a}}$. Hence, $x = a \cdot \sqrt{\frac{b}{a}} = a \cdot \frac{\sqrt{ab}}{a} = \sqrt{ab}$.

. A. 168; B. 45; C. 176; D. 8; E. 2310; F. $2 - (\frac{1}{2})^{23}$; G. k^2;

H. $\frac{k^2 + k}{2}$; I. $\frac{3(1 - 8^k)}{-7}$

. 111. 6. 1368 inches. 7. 850. 8. 24 cm.

. Perimeters: $80 + 40\sqrt{2}$ cm.; Areas: 200 sq. cm.

. 200π cm.

. Prove $\forall_n \sum_{j=1}^{n} (2j - 1) = n^2$

(1) $\sum_{j=1}^{1} (2j - 1) = 2 \cdot 1 - 1 = 1$; $1^2 = 1$.

Hence, the theorem holds for n = 1.

(2) Assume: $\sum_{j=1}^{k}(2j - 1) = k^2$

$$\sum_{j=1}^{k+1}(2j - 1) = \sum_{j=1}^{k}(2j - 1) + 2(k + 1) - 1$$

$$= k^2 + 2(k + 1) - 1$$

$$= k^2 + 2k + 1$$

$$\sum_{j=1}^{k+1}(2j - 1) = (k + 1)^2$$

Hence: $\sum_{j=1}^{k}(2j - 1) = k^2 \Rightarrow \sum_{j=1}^{k+1}(2j - 1) = (k + 1)^2$

(3) Consequently, by (1), (2), and the P.M.I., $\forall_n \sum_{j=1}^{n}(2j - 1) = n^2$

EXERCISE 3

1. A. (1), (2), or (4); B. (1), or (3); C. (1) or (3);
   D. (2); E. (1) F. (2)

2. A. Convergent; Geometric with $|r|<1$; B. Convergent; Geometric with $|r|<1$;
   C. Divergent; Harmonic series; D. Convergent; p-series with $p > 1$.
   E. Convergent; Geometric with $|r|<1$; F. Divergent; Geometric with $|r|>1$;
   G. Convergent; Geometric with $|r|<1$; H. Divergent; p-series with $p<1$.

3. A. 8; B. $\frac{1}{2}$; C. $\frac{7}{9}$; D. 3; E. $\frac{4700}{99}$; F. $\frac{9}{4}$

4. A. (a) $1, \frac{4}{3}, \frac{3}{2}, \frac{8}{5}$ or $\frac{2}{2}, \frac{4}{3}, \frac{6}{4}, \frac{8}{5}$; (b) $\frac{2n}{n+1}$ (c) 2; (d) Conv. (e) 2
   B. (a) $1, \frac{6}{5}, \frac{9}{7}, \frac{4}{3}$ or $\frac{3}{3}, \frac{6}{5}, \frac{9}{7}, \frac{12}{9}$ (b) $\frac{3n}{2n+1}$ (c) $\frac{3}{2}$; (d) Conv. (e) $\frac{3}{2}$
   C. (a) $\frac{5}{2}, \frac{10}{3}, \frac{15}{4}, 4$ or $\frac{5}{2}, \frac{10}{3}, \frac{15}{4}, \frac{20}{5}$ (b) $\frac{5n}{n+1}$ (c) 5, (d) Conv. (e) 5
   D. (a) $\frac{1}{2}, \frac{4}{3}, \frac{9}{4}, \frac{16}{5}$ (b) $\frac{n^2}{n+1}$ (c) ∞ (d) Div. (e) ---
   F. (a) $\frac{3}{2}, 2, \frac{9}{4}, \frac{12}{5}$ or $\frac{3}{2}, \frac{6}{3}, \frac{9}{4}, \frac{12}{5}$ (b) $\frac{3n}{n+1}$ (c) 3 (d) Conv. (e) 3
   G. (a) $\frac{1}{3}, \frac{2}{5}, \frac{3}{7}, \frac{4}{9}$ (b) $\frac{n}{2n+1}$ (c) $\frac{1}{2}$ (d) Conv. (e) $\frac{1}{2}$

   E. (a) $1, \frac{5}{4}, \frac{15}{11}, \frac{10}{7}$ or $\frac{5}{5}, \frac{10}{8}, \frac{15}{11}, \frac{20}{14}$ (b) $\frac{5n}{3n+2}$ (c) $\frac{5}{3}$ (d) conv. (e) $\frac{5}{3}$

GENERAL SUMMARY

To determine whether or not an infinite series is convergent or divergent, check these conditions:

CONVERGENT

The series MAY BE convergent if

if $\lim_{n \to \infty} a_n = 0$

The above is a necessary but not a sufficient condition for convergence.

If $\lim_{n \to \infty} a_n = 0$, then you need to establish one of the following for proof:

1. The series is geometric with $|r| < 1$ (The series has a finite sum.)
2. It is a p-series with $p > 1$.
3. The series is term by term less than a known series which converges.
4. The limit of the sequence of partial sums is some finite number. (This finite number is the sum of the series.)

DIVERGENT

The series is divergent if any one of the following is true:

1. The $\lim_{n \to \infty} a_n \neq 0$
2. The series is geometric with $|r| \geq 1$.
3. The series is a p-series such that $p \leq 1$.
4. The series is term by term larger than a known series which diverges.
5. The series is the harmonic series: $1 + \frac{1}{2} + \frac{1}{3} + \ldots$
6. The limit of the sequence of partial sums of the series tends to infinity.

SEQUENCES AND SERIES

TRIAL RUN

I. Matching. (Give all correct matches. There can be more than one letter choice for each number choice.)

1. $a_n = \frac{1}{n}$
2. $a_n = 2n$
3. $a_n = 3n + 1$
4. $a_n = (\frac{1}{2})^n$
5. $a_n = n!$
6. $a_n = \begin{cases} 8 \text{ for } n \text{ odd} \\ \frac{1}{n} \text{ for } n \text{ even} \end{cases}$
7. $a_n = (-1)^{n-1}(2n - 3)$
8. $a_n = \log(2n + 2)$

A. A convergent sequence
B. A divergent sequence
C. An arithmetic sequence
D. A geometric sequence
E. A monotonic increasing sequence
F. A monotonic decreasing sequence
G. An oscillating sequence
H. $\lim_{n\to\infty} a_n = 0$

II. Write each of the following series using Σ notation:

1. $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots$
2. $.3 + .03 + .003 + \ldots$
3. $\frac{\sqrt{2}}{3} + \frac{\sqrt{3}}{4} + \frac{\sqrt{4}}{5} + \frac{\sqrt{5}}{6} + \ldots$
4. $1 + 2^2 + 3^3 + \ldots + 10^{10}$
5. $\log_2 3 + \log_4 5 + \log_6 7 + \ldots + \log_{20} 21$
6. $1 + 2 + 6 + 24 + 120$
7. $1 - 2 + 3 - 4 + 5 - 6 + 7$.

III. Write the first few terms of each sequence defined below. Determine whether or not the sequence converges. If the sequence converges, tell the number to which it converges.

1. $a_n = \frac{1}{n}$
2. $a_n = \frac{3n + 1}{n - 2}$
3. $a_n = 4(\frac{1}{2})^n$
4. $a_n = \log \frac{10n + 1}{n}$
5. $a_n = (-1)^n (2n)^{-n}$
6. $a_n = \frac{(n + 1)!}{n!}$

IV. Evaluate each of the following:

1. $\lim_{n\to\infty} \frac{3n^2 + 2}{5n^2 - n}$
2. $\lim_{n\to\infty} \frac{1}{n^2}$
3. $\lim_{n\to\infty} \frac{(n + 1)(n - 2)}{(4n - 1)(n)}$
4. $\lim_{n\to\infty} 8\left[\frac{n(n + 2)(n - 2)}{9n^3}\right]$
5. $\lim_{n\to\infty} \frac{n!}{(n + 1)!}$

V. Define: Arithmetic Sequence; Arithmetic Series; Geometric Sequence; Geometric Series.

VI. Evaluate each of the following:

1. Given an arithmetic series with $a_1 = 3$; $n = 50$; and $d = 4$. Find S_{50}.

2. Given a geometric series with $r = 2$; $a_1 = 1$. Find S_{10}.

3. Given an arithmetic series with $a_1 = 13$ and $a_{12} = -15$. Find S_{12}.

4. $\sum_{n=1}^{8} (2n - 1) =$ 5. $\sum_{n=1}^{\infty} 3(\frac{1}{5})^n =$ 6. $\sum_{n=1}^{5} \frac{3n}{n+1} =$

7. $\sum_{n=1}^{\infty} (.4)^n$ 8. $\sum_{n=1}^{10} (-1)^n(2) =$ 9. $\sum_{n=1}^{11} (-1)^n(2) =$

VII. Find the arithmetic mean and the geometric mean for each given pair of numbers

1. 6 and 22; 2. -4 and - 44; 3. 1 and 2; 4. π and 2π; 5. x and y.

VIII. Write the first 15 terms of the Fibonacci Sequence.

IX. 1. Find the sum of the even integers between 102 and 204 inclusive.

2. Find the sum of the first 500 odd integers.

3. A ball is dropped from a height of 4 ft. Each time it strikes the ground it rebounds .75 of the distance it fell. Find the total distance traveled by the ball.

4. Given a square 2" on a side. A new square is inscribed in the given square by joining the midpoints of the given square. This process is then repeated to form another inscribed square. What is the sum of the perimeters of all possible squares formed? What is the sum of the areas of all possible squares formed.

MATH ANALYSIS ONLY

X. Write the first 8 terms of:

1. The harmonic series. 2. A p-series which converges.

3. A p-series which diverges. 4. A geometric series which diverges.

XI. Prove that the positive geometric mean of two positive numbers is less than or equal to the arithmetic mean.

XII. Describe each of the following series as convergent or divergent. Give a reason for your answer.

1. $\frac{1}{2} + \frac{2}{3} + \frac{3}{4} + \ldots + \frac{n}{n+1} + \ldots$ 2. $.1 + .01 + .001 + \ldots + (.1)^{n-1} + \ldots$

3. $1 + \frac{1}{\sqrt[3]{2}} + \ldots + \frac{1}{\sqrt[3]{n}} + \ldots$ 4. $3 + \frac{3}{2} + 1 + \frac{3}{4} + \ldots + \frac{3}{n} + \ldots$

XIII. Determine whether each of the following series converges or diverges. Use the method of the sequence of partial sums. If the series converges determine the number to which the series converges.

1. $\sum_{n=1}^{\infty} \frac{6}{(n+2)(n+3)}$
2. $\sum_{n=1}^{\infty} \frac{-7}{(2n-3)(2n-1)}$
3. $\sum_{n=1}^{\infty} \frac{2n^2-1}{(2n+1)(2n-1)}$

XIV. Prove each of the following using PMI:

1. $\forall_n \sum_{j=1}^{n} \frac{1}{j(j+1)} = \frac{n}{n+1}$
2. $\forall_n \sum_{j=1}^{n} (4n+1) = 2n^2 + 3n$

ANSWERS

I. 1. A., F., H. 2. B., C., E. 3. B., C., E. 4. A., D., F., H
5. B., E. 6. B., G. 7. B., G. 8. B., E.

II. 1. $\sum_{n=1}^{\infty} \frac{1}{n}$ 2. $\sum_{n=1}^{\infty} 3(.1)^n$ 3. $\sum_{n=1}^{\infty} \sqrt{\frac{n+1}{n+2}}$ 4. $\sum_{n=1}^{10} n^n$

5. $\sum_{n=1}^{10} \log_{2n}(2n+1)$ 6. $\sum_{n=1}^{5} n!$ 7. $\sum_{n=1}^{7} (-1)^{n-1}(n)$

III. 1. Converges to 0. 2. Converges to 3; 3. Converges to 0;
4. Converges to 1; 5. Converges to 0; 6. Diverges.

IV. 1. $\frac{3}{5}$; 2. 0; 3. $\frac{1}{4}$; 4. $\frac{8}{9}$; 5. 0.

V. See LAP.

VI. 1. 5050; 2. 1023; 3. -12; 4. 64; 5. $\frac{3}{4}$; 6. 10.65;
7. $\frac{2}{3}$; 8. 0; 9. -2.

VII. 1. 14 and $2\sqrt{33}$; 2. -24 and $-4\sqrt{11}$; 3. $\frac{3}{2}$ and $\sqrt{2}$.
4. $\frac{3\pi}{2}$ and $\pi\sqrt{2}$; 5. $\frac{x+y}{2}$ and $\sqrt{xy}$ for $xy \geq 0$

VIII. 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610

IX. 1. 7956; 2. 250,000; 3. 28 feet;
4. $(16 + 8\sqrt{2})$ inches. 8 sq. in.

X. 1. $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}$

2. Any p-series with $p > 1$.

3. Any p-series with $p \leq 1$.

4. Any geometric series with $|r| \geq 1$.

XI. Given $a \geq 0$ and $b \geq 0$, show $\sqrt{ab} \leq \frac{a + b}{2}$

Assume: $\sqrt{ab} \leq \frac{a + b}{2}$

Then: $2\sqrt{ab} \leq a + b$

$4\,ab \leq a^2 + 2ab + b^2$

$0 \leq a^2 - 2ab + b^2$

$0 \leq (a - b)^2$

Since $(a - b)^2 \geq 0$ for all a and b, reverse steps for proof.

XII. 1. Diverges; $\lim_{n\to\infty} a_n \neq 0$;

2. Converges; geometric with $|r| < 1$.

3. Diverges; p-series with $p < 1$.

4. Diverges; term by term greater than the harmonic series.

XIII. 1. $S_n = \frac{2n}{n + 3}$; Converges to 2.

2. $S_n = \frac{7n}{2n - 1}$; Converges to $\frac{7}{2}$.

3. $S_n = \frac{n^2}{2n + 1}$; Diverges.

MATH ANALYSIS STUDENTS:

Be able to derive the formulas for:

1. The sum of an arithmetic series.
2. The sum of a geometric series.
3. Finding the arithmetic mean between two numbers.
4. Finding the geometric mean between two numbers.