Math Analysis
Trigonometry
Learning Activities Package

W. H. I. P. THEOREMS L.A.P.

Suggested time: 3 weeks

This L.A.P. is called W.H.I.P. Theorems because it concerns itself with theorems that you will find Wild, Handy, Important, and Powerful. (It is also called WHIP because it might be painful and tedious.) However, when you have mastered it you will experience the usual sheer delight!

BEHAVIORAL OBJECTIVES

I. Memorize each of the following trigonometric theorems:

A. The even function relationships
1. $\cos(-\theta) = \cos\theta$
2. $\sec(-\theta) = \sec\theta$

B. The odd function relationships
1. $\sin(-\theta) = -\sin\theta$
2. $\tan(-\theta) = -\tan\theta$
3. $\csc(-\theta) = -\csc\theta$
4. $\cot(-\theta) = -\cot\theta$

C. The Pythagorean Identity: $\sin^2\theta + \cos^2\theta = 1$

D. The addition theorem: $\cos(\theta + \phi) = \cos\theta\cos\phi - \sin\theta\sin\phi$

E. The half number theorems
1. $\sin\frac{\theta}{2} = \pm\sqrt{\frac{1-\cos\theta}{2}}$
2. $\cos\frac{\theta}{2} = \pm\sqrt{\frac{1+\cos\theta}{2}}$

II. Be able to prove each of the following trigonometric theorems

A. The Pythagorean identities
1. $1 + \cot^2\theta = \csc^2\theta$
2. $\tan^2\theta + 1 = \sec^2\theta$

* B. The half-number theorem: $\tan\frac{\theta}{2} = \pm\frac{1-\cos\theta}{\sin\theta}$

C. The addition theorems
1. $\sin(\theta \pm \phi) = \sin\theta\cos\phi \pm \cos\theta\sin\phi$
2. $\cos(\theta - \phi) = \cos\theta\cos\phi + \sin\theta\sin\phi$
*3. $\tan(\theta \pm \phi) = \frac{\tan\theta \pm \tan\phi}{1 \mp \tan\theta\tan\phi}$

D. Supplementary number theorems
1. $\sin(\pi \pm \theta) = \mp\sin\theta$
2. $\cos(\pi \pm \theta) = -\cos\theta$
3. $\tan(\pi \pm \theta) = \mp\tan\theta$

*Optional for Trigonometry course. Memorize them though you need not know how to prove them.

E. Complementary number theorems

1. $\sin\left(\frac{\pi}{2} \pm \theta\right) = \cos\theta$

2. $\cos\left(\frac{\pi}{2} \pm \theta\right) = \mp \sin\theta$

*3. $\tan\left(\frac{\pi}{2} \pm \theta\right) = \mp \cot\theta$

*F. The double number theorems

1. $\sin 2\theta = 2\sin\theta\cos\theta$

2. $\cos 2\theta = \cos^2\theta - \sin^2\theta$

3. $\cos 2\theta = 1 - 2\sin^2\theta$

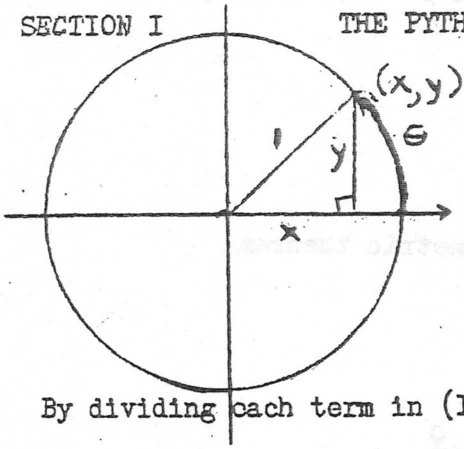
4. $\cos 2\theta = 2\cos^2\theta - 1$

5. $\tan 2\theta = \dfrac{2\tan\theta}{1 - \tan^2\theta}$

III. Be able to use the basic trigonometric theorems to solve problems as in Section 8.

P.S. The more of the basic trigonometric theorems that you memorize the freer you are to procede. Continue to become more and more at home with them.

SECTION I THE PYTHAGOREAN IDENTITIES



For any ordered pari (x,y) on the unit circle by the Pythagorean theorem:

$$x^2 + y^2 = 1$$

Hence, it follows that:

$$(\sin\theta)^2 + (\cos\theta)^2 = 1$$

Or, in customary notation:

$$\sin^2\theta + \cos^2\theta = 1 \quad (1)$$

By dividing each term in (1) by $\sin^2\theta$ it follows that:

$$1 + \cot^2\theta = \csc^2\theta \quad \text{for } \theta \neq k\pi \quad (2)$$

By dividing each term in (1) by $\cos^2\theta$ it follows that:

$$\tan^2\theta + 1 = \sec^2\theta; \quad \text{for } \theta \neq \frac{(2k+1)\pi}{2} \quad (3)$$

ASSIGNMENT:

1. Memorize (1)

2. Be able to derive (2) and (3) from (1). Include restrictions, and know why they are important.

3. Identify each of the following as true or false for all real values of x for which the functions are defined:

(a) $\csc^2 x - 1 = \cot^2 x$ (b) $1 - \sin^2 x = \cos^2 x$

(c) $\sin^2 x + \cos^2 x = \csc^2 x - \cot^2 x$ (d) $1 - \csc^2 x = \cot^2 x$

(e) $2\sin^2 x + \cos^2 x = 2$

(f) $\csc^2 x - \sec^2 x = \cot^2 x - \tan^2 x$

(g) $\sec^2 x - \tan^2 x = 1$

(h) $\sin^2 x + \cos^2 x + \cot^2 x = \csc^2 x$

(i) $\sin x + \cos x = 1$

(j) $2 + \cot^2 x + \tan^2 x = \csc^2 x + \sec^2 x$

(k) $\cos^2 x - 1 = \sin^2 x$

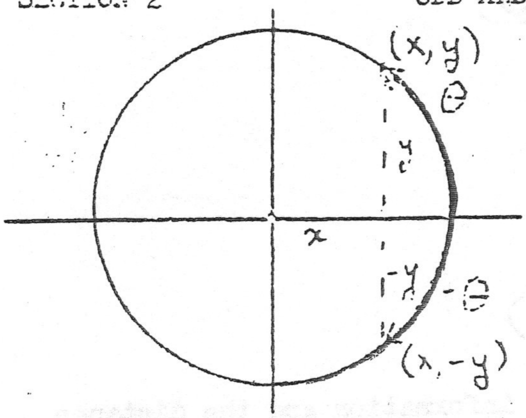
(l) $\sin^2 x - 1 = -\cos^2 x$

(m) $\sin^4 x + 2\sin^2 x\cos^2 x + \cos^4 x = 1$

(n) $\tan^2 x - \sec^2 x = \csc^2 x - \cot^2 x$

(o) $\sec^2 x - \tan^2 x = 2 - \sin^2 x - \cos^2 x$

SECTION 2 ODD AND EVEN FUNCTIONS



Study the diagram on the left.

Notice:

$\sin(-\theta) = -\sin\theta$		(4)
$\cos(-\theta) = \cos\theta$		(5)
$\tan(-\theta) = -\tan\theta$		(6)
$\csc(-\theta) = -\csc\theta$		(7)
$\sec(-\theta) = \sec\theta$		(8)
$\cot(-\theta) = -\cot\theta$		(9)

If $f(-x) = -f(x)$, then f is an odd function. Which of the trig functions are odd?

If $f(-x) = f(x)$, then f is an even function. Which of the trig functions are even?

ASSIGNMENT:

1. Be able to identify each trig function as either even or odd.
2. Know theorems (4) - (9) in your sleep. Also know them when you are awake!
3. Rewrite each of the following as a function of θ where $0 \le \theta \le \frac{\pi}{2}$.

   eg. $\tan(-\frac{3\pi}{4}) = \tan\frac{\pi}{4}$

   (a) $\sin(-\frac{\pi}{2})$ (b) $\cos(\frac{4\pi}{3})$ (c) $\sec(\frac{7\pi}{6})$

   (d) $\csc(-\pi)$ (e) $\cot(\frac{5\pi}{6})$ (f) $\tan(-\frac{5\pi}{3})$

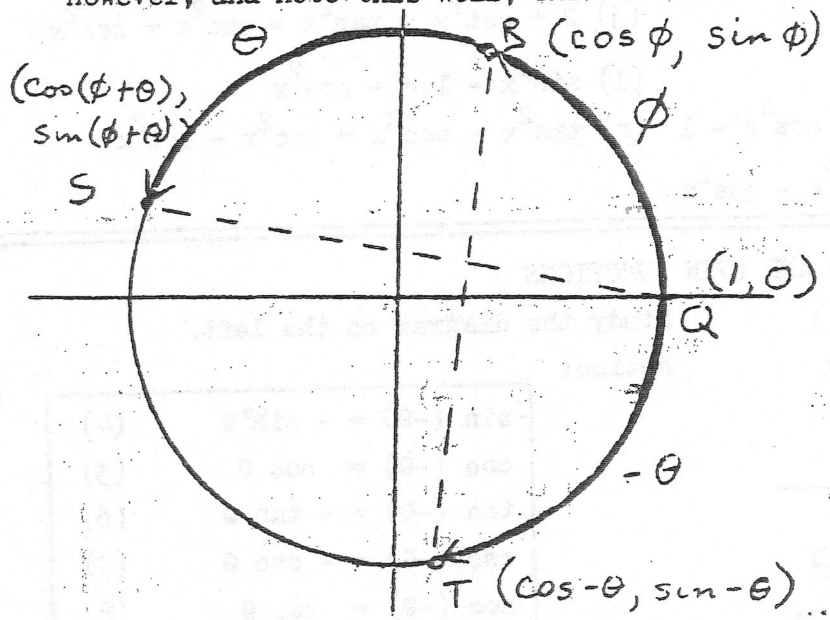
   (g) $\cot(-\frac{\pi}{6})$ (h) $\tan(-\frac{5\pi}{4})$ (i) $\sec(-\frac{\pi}{4})$

4. What kind of symmetry is present in the graph of an odd function?
5. What kind of symmetry is present in the graph of an even function?
6. How many of the six trig functions are even?

SECTION 3 ADDITION AND SUBTRACTION FORMULAS

It would be nice if for all numbers θ and ϕ, $\cos(\theta + \phi) = \cos\theta + \cos\phi$.

However, and note this well, THE ABOVE IS NOT THE CASE!!!



In the above diagram, $\overline{SQ} \cong \overline{RT}$. Using this information and the distance formula we have:

$$\sqrt{(\cos(\phi + \theta) - 1)^2 + (\sin(\phi + \theta) - 0)^2} = \sqrt{(\cos\phi - \cos(-\theta))^2 + (\sin\phi - \sin(-\theta))^2}$$

Squaring both sides and expanding the binomials we have:

$$\cos^2(\phi + \theta) - 2\cos(\phi + \theta) + 1 + \sin^2(\phi + \theta) = \cos^2\phi - 2\cos\phi\cos(-\theta) + \cos^2(-\theta) + \sin^2\phi - 2\sin\phi\sin(-\theta) + \sin^2(-\theta).$$

$-2\cos(\phi + \theta) + 2 = -2\cos\phi\cos\theta + 2\sin\phi\,\text{cin}\,\theta + 2$ [Theorem (1) and (4)]

$-2\cos(\phi + \theta) = -2\cos\phi\cos\theta + 2\sin\phi\sin\theta$ [Algebra]

$\cos(\phi + \theta) = \cos\phi\cos\theta - \sin\phi\sin\theta$ [Algebra]

THEOREM 10

$$\cos(\phi + \theta) = \cos\phi\cos\theta - \sin\phi\sin\theta$$

Theorem 10 is the only theorem that requires such a geometric proof. The other proofs in this section can be done algebraically. To prove the theorems of this section we call on previously proved theorems and algebra.

Sample: Prove: $\cos(\phi - \theta) = \cos\phi\cos\theta + \sin\phi\sin\theta$ Theorem 11

Proof: $\cos(\phi - \theta) = \cos(\phi + (-\theta))$ Algebra

$= \cos\phi\cos(-\theta) - \sin\phi\sin(-\theta)$ [Theorem 10]

$= \cos\phi\cos\theta + \sin\phi\sin\theta$ [Theorems 5, 4, and algebra]

ASSIGNMENT:

Be able to prove each of the following theorems. Memorize the theorems too. It is important to memorize them to procede quickly with your work. It is your important to know how to prove them so that you can work things out should your memory fail.

1. Theorem 12: $\cos(\frac{\pi}{2} - \theta) = \sin\theta$

2. Theorem 13: $\sin(\phi + \theta) = \sin\phi\cos\theta + \cos\phi\sin\theta$

   Hint: $\sin(\phi + \theta) = \cos(\frac{\pi}{2} - (\phi + \theta))$
   $= \cos((\frac{\pi}{2} - \phi) - \theta)$

3. Theorem 14: $\sin(\phi - \theta) = \sin\phi\cos\theta - \cos\phi\sin\theta$

4. Theorem 15: $\tan(\phi + \theta) = \dfrac{\tan\phi + \tan\theta}{1 - \tan\phi\tan\theta}$

   Hint: $\tan(\phi + \theta) = \dfrac{\sin(\phi + \theta)}{\cos(\phi + \theta)}$

5. Theorem 16: $\tan(\phi - \theta) = \dfrac{\tan\phi - \tan\theta}{1 + \tan\phi\tan\theta}$

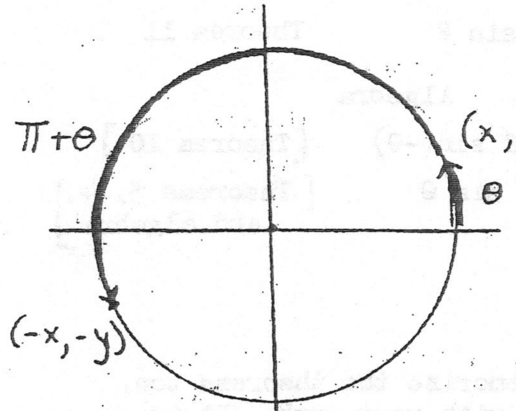
* 6. Theorem 17: $\cot(\phi + \theta) = \dfrac{\cot\phi\cot\theta - 1}{\cot\theta + \cot\phi}$

* 7. Theorem 18: $\cot(\phi - \theta) = \dfrac{\cot\phi\cot\theta + 1}{\cot\theta - \cot\phi}$

Show your proofs to your teacher to make sure that you are doing them in proper form. *Optional for Trigonometry students.

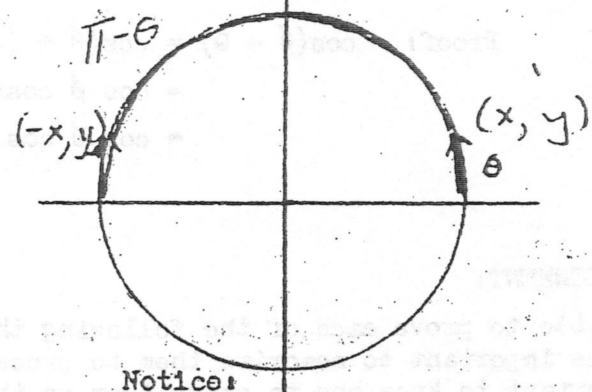
SECTION 4 SUPPLEMENTARY NUMBER THEOREMS

The supplementary number theorems deal with adding or subtracting a number, θ, to/from π. Study the diagrams below and note what happens geometrically.



Notice:

$\sin(\pi + \theta) = -\sin\theta$
$\cos(\pi + \theta) = -\cos\theta$
$\tan(\pi + \theta) = \tan\theta$



Notice:

$\sin(\pi - \theta) = \sin\theta$
$\cos(\pi - \theta) = -\cos\theta$
$\tan(\pi - \theta) = -\tan\theta$

A single example, a diagram is not a proof. The above can be proved algebraically by using theorems already proved and the facts that $\sin\pi = 0$ and $\cos\pi = -1$.

ASSIGNMENT:

Prove each of the following theorems:

1. Theorem 19: $\sin(\pi + \theta) = -\sin\theta$
2. Theorem 20: $\cos(\pi + \theta) = -\cos\theta$
3. Theorem 21: $\tan(\pi + \theta) = \tan\theta$
4. Theorem 22: $\sin(\pi - \theta) = \sin\theta$
5. Theorem 23: $\cos(\pi - \theta) = -\cos\theta$
6. Theorem 24: $\tan(\pi - \theta) = -\tan\theta$

SECTION 5 COMPLEMENTARY NUMBER THEOREMS

By using the same algebraic technique as you did in Section 3, prove the theorems of this section. Note: $\sin\frac{\pi}{2} = 1$ and $\cos\frac{\pi}{2} = 0$.

ASSIGNMENT:

Prove each of the following theorems:

1. Theorem 25: $\sin(\frac{\pi}{2} + \theta) = \cos\theta$
2. Theorem 26: $\cos(\frac{\pi}{2} + \theta) = -\sin\theta$
3. Theorem 27: $\tan(\frac{\pi}{2} + \theta) = -\cot\theta$

4. Theorem 28: $\sin(\frac{\pi}{2} - \theta) = \cos\theta$

5. Theorem 29: $\cos(\frac{\pi}{2} - \theta) = \sin\theta$

6. Theorem 30: $\tan(\frac{\pi}{2} - \theta) = \cot\theta$

SECTION 6 DOUBLE NUMBER THEOREMS

In this section we again call on algebra and previously proved theorems to derive theorems for $\sin 2\theta$, $\cos 2\theta$, and $\tan 2\theta$. Procede at once to the assignment. Show your proofs to your teacher.

ASSIGNMENT:

Prove each of the following theorems:

1. Theorem 31: $\sin 2\theta = 2\sin\theta\cos\theta$
2. Theorem 32: $\cos 2\theta = \cos^2\theta - \sin^2\theta$
3. Theorem 33: $\cos 2\theta = 1 - 2\sin^2\theta$
4. Theorem 34: $\cos 2\theta = 2\cos^2\theta - 1$

* 5. Theorem 35: $\tan 2\theta = \dfrac{2\tan\theta}{1 - \tan^2\theta}$ (optional for trigonometry students)

SECTION 7 HALF-NUMBER THEOREMS

This section concerns itself with three final theorems for this L.A.P. They deal with $\sin\frac{\theta}{2}$, $\cos\frac{\theta}{2}$, and $\tan\frac{\theta}{2}$. Procede as you did in the previous section. Do the proofs and show them to your teacher. It is sufficient that Trigonometry students memorize the theorems.

ASSIGNMENT:

Prove each of the following theorems:

1. Theorem 36: $\sin\frac{\theta}{2} = \pm\sqrt{\dfrac{1 - \cos\theta}{2}}$

   Hint: $\cos 2\left(\frac{\theta}{2}\right) = 1 - 2\sin^2\frac{\theta}{2}$

2. Theorem 37: $\cos\frac{\theta}{2} = \pm\sqrt{\dfrac{1 + \cos\theta}{2}}$

3. Theorem 38: $\tan\frac{\theta}{2} = \pm\dfrac{1 - \cos\theta}{\sin\theta}$

NOTE: For each of the above theorems, the choice of "+" or "-" will depend upon the quadrant placement of $\theta/2$. That sign choice you will have to make when you work a particular problem.

SECTION 8 APPLICATION OF THE TRIG THEOREMS

This section deals solely with the use of the theorems developed in this L.A.P. Do the following assignment carefully.

ASSIGNMENT:

1. Evaluate each of the following:

(a) $\sin(\frac{\pi}{2} - \frac{\pi}{6})$ (b) $\cos(\frac{\pi}{3} + \frac{\pi}{4}) =$

(c) $\sin(\frac{\pi}{4} - \frac{\pi}{3}) =$ (d) $\cos\frac{\pi}{12} =$

(e) $\tan(\frac{\pi}{6} + \frac{\pi}{4}) =$

2. If tan θ is less than 0, and $\cos\theta = \frac{3}{5}$, find the value of the six trig functions of θ. Remember that $\sin^2\theta + \cos^2\theta = 1$.

3. For $\pi < \theta < \frac{3\pi}{2}$ and $\cos\theta = -\frac{4}{5}$, and for $\frac{\pi}{2} < \phi < \pi$ and $\tan\phi = -\frac{5}{12}$, find:

(a) $\sin(\theta + \phi)$ (b) $\cos(\theta + \phi)$ (c) $\sin(\theta - \phi)$

(d) $\cos(\theta - \phi)$ (e) $\tan(\theta + \phi)$ (f) $\tan(\theta - \phi)$

(g) $\csc(\theta + \phi)$ (h) $\sec(\theta + \phi)$ (i) $\cot(\theta - \phi)$

4. Find the value of sin 2θ, cos 2θ and tan 2θ for the following:

(a) $\cos\theta = \frac{3}{5}$ and $0 < \theta < \frac{\pi}{2}$ (b) $\tan\theta = \frac{4}{3}$ and $\pi < \theta < \frac{3\pi}{2}$

(c) $\sin\theta = \frac{2}{3}$ and $\frac{\pi}{2} < \theta < \pi$ (d) $\cos\theta = \frac{3}{4}$ and $\frac{3\pi}{2} < \theta < 2\pi$

5. Find the value of $\sin\frac{\theta}{2}$, $\cos\frac{\theta}{2}$, and $\tan\frac{\theta}{2}$ for the following:

(a) $\cos\theta = \frac{1}{2}$ and $\frac{3\pi}{2} < \theta < 2\pi$ (b) $\sin\theta = \frac{-3}{5}$ and $\pi < \theta < \frac{3\pi}{2}$

(c) $\cos\theta = -\frac{5}{12}$ and $\frac{\pi}{2} < \theta < \pi$ (d) $\sin\theta = \frac{12}{13}$ and $0 < \theta < \frac{\pi}{2}$

6. Derive a formula for sin 3θ in terms of sin θ

7. Derive a formula for cos 3θ in terms of cos θ.

SECTION 9 EVALUATION

1. Review the Behavioral Objectives. Memorize as many theorems as you can.
2. Practice deriving the theorems as indicated in the Behavioral Objectives.
3. Take the Trial Run for this L.A.P.
4. Take the test on this L.A.P.

ANSWERS

SECTION 1

3. (a) T; (b) T; (c) T; (d) F; (e) F; (f) T; (g) T;
(h) T; (i) F; (j) T; (k) F; (l) T; (m) T; (n) F;
(o) T.

SECTION 2

3. (a) $-\sin\frac{\pi}{2}$; (b) $-\cos\frac{\pi}{3}$; (c) $-\sec\frac{\pi}{6}$;
(d) undefined; (e) $-\cot\frac{\pi}{6}$; (f) $\tan\frac{\pi}{3}$;
(g) $-\cot\frac{\pi}{6}$; (h) $-\tan\frac{\pi}{4}$; (i) $\sec\frac{\pi}{4}$

4. origin; 5. Y-axis; 6. two

SECTION 8

1. (a) $\frac{\sqrt{3}}{2}$; (b) $\frac{\sqrt{2}-\sqrt{6}}{4}$; (c) $\frac{\sqrt{2}-\sqrt{6}}{4}$;
(d) $\frac{\sqrt{2}+\sqrt{6}}{4}$; (e) $2+\sqrt{3}$.

2. $\sin\theta = -\frac{4}{5}$; $\tan\theta = -\frac{4}{3}$; $\sec\theta = \frac{5}{3}$; $\csc\theta = -\frac{5}{4}$; $\cot\theta = -\frac{3}{4}$.

3. (a) $\frac{16}{65}$; (b) $\frac{63}{65}$; (c) $\frac{56}{65}$; (d) $\frac{33}{65}$; (e) $\frac{16}{63}$
(f) $\frac{56}{33}$ (g) $\frac{65}{16}$; (h) $\frac{65}{63}$; (i) $\frac{33}{56}$

4. (a) $\frac{24}{25}$; $\frac{-7}{25}$; $-\frac{24}{7}$; (b) $\frac{24}{25}$; $\frac{-7}{25}$; $-\frac{24}{7}$;
(c) $\frac{-4\sqrt{5}}{9}$; $\frac{1}{9}$; $-4\sqrt{5}$; (d) $\frac{-3\sqrt{7}}{8}$; $\frac{1}{8}$; $-3\sqrt{7}$

5. (a) $\frac{1}{2}$; $-\frac{\sqrt{3}}{2}$; $-\frac{1}{\sqrt{3}}$ (b) $\frac{3}{\sqrt{10}}$; $\frac{-1}{\sqrt{10}}$; -3
(c) $\sqrt{\frac{17}{24}}$; $\sqrt{\frac{7}{24}}$; $\sqrt{\frac{17}{7}}$ (d) $\frac{2}{\sqrt{13}}$; $\frac{3}{\sqrt{13}}$; $\frac{2}{3}$

6. $\sin 3\theta = 3\sin\theta - 4\sin^3\theta$

7. $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$

MATH ANALYSIS W.H.I.P. LAP TRIAL RUN

I. COMPLETE EACH OF THE FOLLOWING:

1. $\sin^2\theta =$ ________
2. $\sin(-\theta) =$ ________
3. $\cos(-\theta) =$ ________
4. $\tan(-\theta) =$ ________
5. $\csc(-\theta) =$ ________
6. $\sec(-\theta) =$ ________
7. $\cot(-\theta) =$ ________
8. $\cos(\emptyset + \theta) =$ ________
9. $\cos(\emptyset - \theta) =$ ________
10. $\cos(\frac{\pi}{2} - \theta) =$ ________
11. $\sin(\emptyset + \theta) =$ ________
12. $\sin(\emptyset - \theta) =$ ________
13. $\tan(\emptyset + \theta) =$ ________
14. $\tan(\emptyset - \theta) =$ ________
15. $\sin(\pi - \theta) =$ ________
16. $\cos(\pi - \theta) =$ ________
17. $\tan(\pi - \theta) =$ ________
18. $\sin(\pi + \theta) =$ ________
19. $\cos(\pi + \theta) =$ ________
20. $\tan(\pi + \theta) =$ ________
21. $\sin(\frac{\pi}{2} - \theta) =$ ________
22. $\cos(\frac{\pi}{2} - \theta) =$ ________
23. $\tan(\frac{\pi}{2} - \theta) =$ ________
24. $\sin(\frac{\pi}{2} + \theta) =$ ________
25. $\cos(\frac{\pi}{2} + \theta) =$ ________
26. $\tan(\frac{\pi}{2} + \theta) =$ ________
27. $\sin 2\theta =$ ________
28. $\cos 2\theta =$ ________ ; or ________ ; or ________
29. $\tan 2\theta =$ ________
30. $\sin\frac{\theta}{2} =$ ________
31. $\cos\frac{\theta}{2} =$ ________
32. $\tan\frac{\theta}{2} =$ ________
33. $\csc^2\theta =$ ________
34. $\sec^2\theta =$ ________

II. Be able to prove any of the above theorems.

III. Let θ be an arc in standard position whose endpoint is in Quadrant II. $\sin\theta = \frac{3}{5}$

$\emptyset$ is an arc in standard position with endpoint in Quadrant III. $\tan\emptyset = \frac{12}{5}$.

Find:

1. $\sin(\emptyset + \theta)$
2. $\cos(\emptyset + \theta)$
3. $\tan(\emptyset + \theta)$
4. $\sin(\emptyset - \theta)$
5. $\cos(\emptyset - \theta)$
6. $\tan(\emptyset - \theta)$
7. $\sin 2\theta$
8. $\cos 2\theta$
9. $\tan 2\theta$
10. $\sin\frac{\theta}{2}$
11. $\cos\frac{\theta}{2}$
12. $\tan\frac{\theta}{2}$

IV. Complete each of the following as a function of θ.

1. $\tan(\pi + \theta)$
2. $\tan(\frac{\pi}{2} - \theta)$
3. $\cos(\frac{\pi}{2} + \theta)$
4. $\sin(\frac{\pi}{2} - \theta)$
5. $\sin(\pi + \theta)$
6. $\cos(\pi + \theta)$
7. $\sin(\pi - \theta)$
8. $\tan(-\theta)$
9. $\cos(-\theta)$

V. Write each of the following as a function of θ where $0 \le \theta \le \frac{\pi}{2}$.

1. $\sin \frac{5\pi}{4}$ 2. $\cos \frac{5\pi}{6}$ 3. $\sin \frac{2\pi}{3}$ 4. $\sin \frac{5\pi}{3}$

5. $\tan -\frac{5\pi}{4}$ 6. $\sin \frac{4\pi}{3}$ 7. $\tan -\frac{7\pi}{6}$ 8. $\cos \frac{7\pi}{6}$

9. $\sin \frac{3\pi}{4}$ 10. $\cos -\frac{2\pi}{3}$

ANSWERS

I. 1. $1 - \cos^2\theta$ 2. $-\sin\theta$ 3. $\cos\theta$ 4. $-\tan\theta$

5. $-\csc\theta$ 6. $\sec\theta$ 7. $-\cot\theta$ 8. $\cos\phi\cos\theta - \sin\phi\sin\theta$

9. $\cos\phi\cos\theta + \sin\phi\sin\theta$ 10. $\sin\theta$ 11. $\sin\phi\cos\theta + \cos\phi\sin\theta$

12. $\sin\phi\cos\theta - \cos\phi\sin\theta$ 13. $\frac{\tan\phi + \tan\theta}{1 - \tan\phi\tan\theta}$ 14. $\frac{\tan\phi - \tan\theta}{1 + \tan\phi\tan\theta}$

15. $\sin\theta$ 16. $-\cos\theta$ 17. $-\tan\theta$ 18. $-\sin\theta$

19. $-\cos\theta$ 20. $\tan\theta$ 21. $\cos\theta$ 22. $\sin\theta$

23. $\cot\theta$ 24. $\cos\theta$ 25. $-\sin\theta$ 26. $-\cot\theta$

27. $2\sin\theta\cos\theta$ 28. $\cos^2\theta - \sin^2\theta$ or $1 - 2\sin^2\theta$ or $2\cos^2\theta - 1$

29. $\frac{2\tan\theta}{1 - \tan^2\theta}$ 30. $\pm\sqrt{\frac{1 - \cos\theta}{2}}$ 31. $\pm\sqrt{\frac{1 + \cos\theta}{2}}$

32. $\frac{\sin\theta}{1 + \cos\theta}$ or $\frac{1 - \cos\theta}{\sin\theta}$ 33. $1 + \cot^2\theta$ 34. $1 + \tan^2\theta$

II. See LAP

III. $\sin\theta = \frac{3}{5}$, $\cos\theta = -\frac{4}{5}$, $\tan\theta = -\frac{3}{4}$; $\sin\phi = -\frac{12}{13}$, $\cos\phi = -\frac{5}{13}$, $\tan\phi = \frac{12}{5}$

1. $\frac{33}{65}$ 2. $\frac{56}{65}$ 3. $\frac{33}{56}$ 4. $\frac{63}{65}$ 5. $-\frac{16}{65}$ 6. $-\frac{63}{16}$

7. $-\frac{24}{25}$ 8. $\frac{7}{25}$ 9. $-\frac{24}{7}$ 10. $\frac{3\sqrt{10}}{10}$ 11. $\frac{\sqrt{10}}{10}$ 12. 3

IV.

1. $\tan\theta$ 2. $\cot\theta$ 3. $-\sin\theta$ 4. $\cos\theta$ 5. $-\sin\theta$ 6. $-\cos\theta$

7. $\sin\theta$ 8. $-\tan\theta$ 9. $\cos\theta$

V.

1 - $\sin\frac{\pi}{4}$ 2. $-\cos\frac{\pi}{3}$ 3. $\sin\frac{\pi}{3}$ 4. $-\sin\frac{\pi}{3}$

5. $-\tan\frac{\pi}{4}$ 6. $-\sin\frac{\pi}{3}$ 7. $-\tan\frac{\pi}{6}$ 8. $-\cos\frac{\pi}{6}$

9. $\sin\frac{\pi}{4}$ 10. $-\cos\frac{\pi}{3}$