BEHAVIORAL OBJECTIVES

I. Define hyperbola as

A. A locus of points from
   1. Two fixed points
   2. A point and a line

B. A conic section

II. Identify the

A. Vertices of a hyperbola
B. Foci of a hyperbola
C. Transverse axis of a hyperbola
D. Conjugate axis of a hyperbola
E. Directrix of a hyperbola
F. Eccentricity of a hyperbola
G. Latera recta of a hyperbola
H. Asymptotes of a hyperbola

III. Given the equation of a hyperbola determine

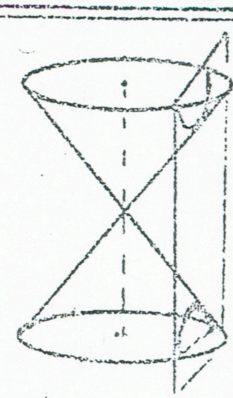
A. The coordinates of the vertices of the hyperbola
B. The coordinates of the foci of the hyperbola
C. The equations of the directrices of the hyperbola
D. The equations of the asymptotes of the hyperbola
E. The length of
   1. The transverse axis
   2. The conjugate axis
   3. Latus rectum
F. The eccentricity of the hyperbola
G. The equation of its conjugate hyperbola
H. Its graph

IV. Determine the equation of a hyperbola given

A. The coordinates of the vertices and the foci
B. The equations of the asymptotes and the vertices
C. The coordinates of the foci and the length of the transvers axis
D. The coordinates of one focus, the equation of a directrix and the eccentricity
E. The coordinates of the endpoints of both axes

V. Solve problems involving the properties of the hyperbola

SECTION I HYPERBOLA--DEFINITION



Hyperbola-- Definition 1: A hyperbola is the intersection of a plane and a right circular double-napped cone such that the plane is parallel to the axis of the cone.

The above definition will not be used algebraically. We include this definition to establish the hyperbola as a conic section.

Hyperbola--Definition 2: A hyperbola is the locus of points in a plane such that the difference from two fixed points to any point of the locus is constant.

Example: Write the equation of the hyperbola with foci (5,0) and (-5,0) and with constant difference 6.

Solution: Choose an arbitrary point (x,y) on the curve. Then the equation becomes:

$$\sqrt{(x-5)^2 + y^2} - \sqrt{(x+5)^2 + y^2} = 6$$

$$\sqrt{(x-5)^2 + y^2} = 6 + \sqrt{(x+5)^2 + y^2}$$

Square both sides and simplify to

$$-5x - 9 = 3\sqrt{(x+5)^2 + y^2}$$

Square both sides and simplify again to

$$\boxed{\frac{x^2}{9} - \frac{y^2}{16} = 1}$$

The center of the above hyperbola is (0,0). To sketch the graph notice that there can be no Y-intercepts as x can never equal zero. In face the smallest positive value that x can assume is 3. The vertices of the hyperbola are (3,0) and (-3,0). The curve is symmetric with respect to the Y-axis.

Taking the equation and solving for y^2 we have:

$$y^2 = 16\left(\frac{x^2}{9} - 1\right)$$

$$y = \pm 4\sqrt{\frac{x^2}{9} - 1}$$

As x becomes very large subtracting 1 doesn't make much difference. It is a little like Rockefeller losing a dollar. Would he ever notice?

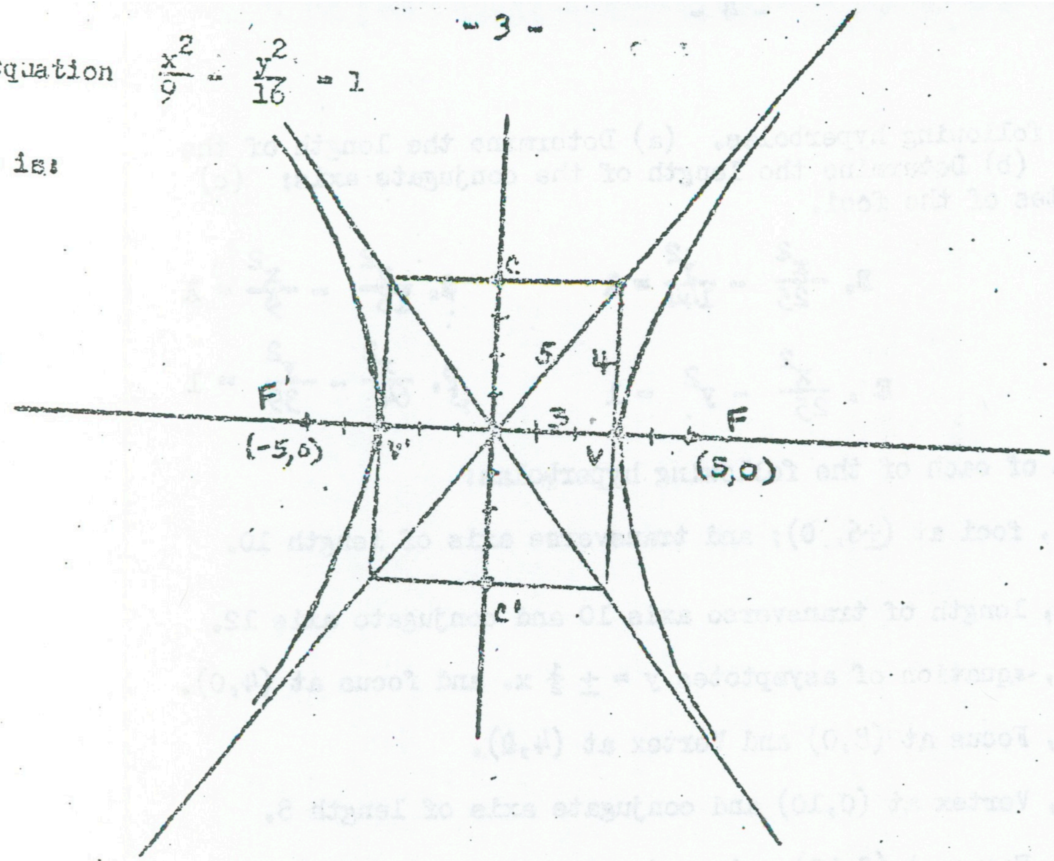
So, as x becomes large y tends to $\pm \frac{4}{3}x$.

These two lines $y = \frac{4}{3}x$ and $y = -\frac{4}{3}x$ are asymptotes of the curve.

On the following page the curve is sketched. Study it carefully.

the equation $\frac{x^2}{9} - \frac{y^2}{16} = 1$

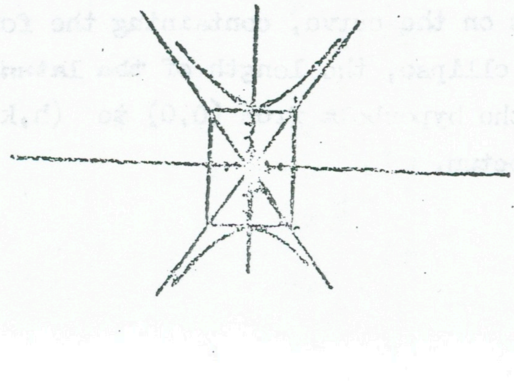
graph is:



0) and V' (-3,0) are the vertices. VV' is called the transverse axis.
is called the conjugate acis. F(5,0) and F' (-5,0) are the foci.
'istance from the center to the focus is customarily called c.
semi-transverse axis is usually called a and the semi-conjugate axis is
ally called b. Notice: $a^2 + b^2 = c^2$. Following this also notice the
ation of the asymptotes is $y = \pm \frac{b}{a} x$.

ph a hyperbola first locate the center. Then sketch the rectangle as
above. Draw the diagonals of the rectangle and extend them to sketch the
otes of the curve. Gently sketch in the curve.

e given special equation, the curve defined by the equation
$- \frac{x^2}{9} = 1$ is called its conjugate. It uses the same center and
symptotes but is a vertical rather that a horizontal hyperbola.



Exercise 1

1. Graph each of the following hyperbolas. (a) Determine the length of the transverse axis; (b) Determine the length of the conjugate axis; (c) Find the coordinates of the foci.

A. $\frac{x^2}{16} - \frac{y^2}{9} = 1$ B. $\frac{x^2}{25} - \frac{y^2}{144} = 1$ C. $\frac{y^2}{16} - \frac{x^2}{9} = 1$

D. $\frac{y^2}{49} - \frac{x^2}{4} = 1$ E. $\frac{x^2}{25} - y^2 = 1$ F. $\frac{x^2}{64} - \frac{y^2}{39} = 1$

2. Write the equation of each of the following hyperbolas:

A. Center at (0,0), foci at ($\pm$6, 0); and transverse axis of length 10.

B. Center at (0,0), length of transverse axis 10 and conjugate axis 12.

C. Cneter at (0,0), equation of asymptotes $y = \pm \frac{1}{2} x$, and focus at (4,0).

D. Center at (0,0), Focus at (8,0) and Vertex at (4,0).

E. Center at (0,0), Vertex at (0,10) and conjugate axis of length 8.

F. Center at (0,0), Focus at (0,10) and conjugate axis of length 8.

For each of the following hyperbolas, write the equation of its conjugate hyperbola:

A. $x^2 - y^2 = 1$ B. $\frac{x^2}{2} - y^2 = 1$ C. $\frac{y^2}{6} - x^2 = 1$

CTION II HYPERBOLA WITH CENTER (h,k)

ging the center from (0,0) to (h,k) simply moves each x component h units and ves each y component k units. The equation in information form becomes:

$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$$

asymptotes will continue to have a slope of $\pm\frac{b}{a}$. However with center at (h,k) lines must contain (h,k). Hence the equations of the asymptotes are

$$\frac{y-k}{x-h} = \pm\frac{b}{a}$$

in the ellipse, the hyperbola has two latera recta. The latus rectum is a ment with endpoints on the curve, containing the focus and perpendiculat to axis. As in the ellipse, the length of the latus rectum is $2b^2/a$. ing the center of the hyperbola from (0,0) to (h,k) does not change the gth of the latus rectum.

EXERCISE 2

. Sketch the graph of each of the following. Determine (a) the coordinates of the vertices; (b) The coordinates of the foci; (c) The equations of the asymptotes. (d) The length of the latus rectum.

A. $\frac{(x-2)^2}{16} - \frac{(y-3)^2}{4} = 1$ B. $\frac{(y+1)^2}{9} - \frac{(x+5)^2}{9} = 1$

C. $\frac{(x-4)^2}{4} - (y+6)^2 = 1$ D. $(x-3)^2 - 5(y+3)^2 = 125$

Write the equation of the hyperbola

A. With center (2,3); transverse axis 12units; conjugate axis 4 units; with transverse axis horizontal.

B. Asymptotes $y = 5x$ and $y = -5x + 10$; vertex at (1,10).

C. Center (3,4); Vertex (5,4) and Focus (6,4).

D. Center (3,4); Vertex (3,6) and Focus (3,7).

E. The length of the latus rectum is 4 units. Center at (3,3) and contains the point (0,0).

F. Vertices at (-1,4) and (-1,6) and foci at (-1,3) and (-1,7).

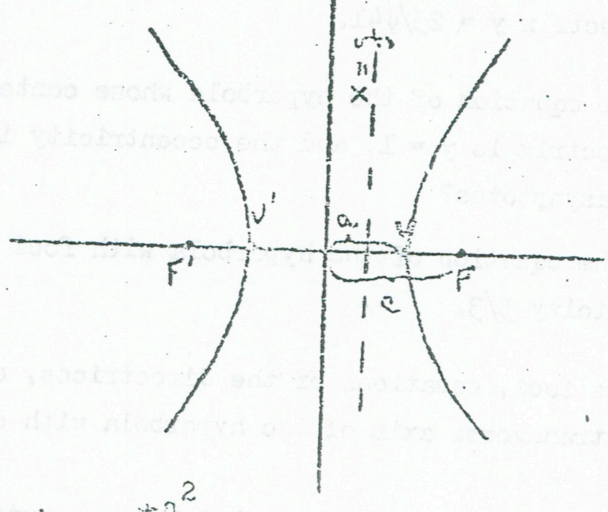
Change each of the following equations into the informational form:

. $4x^2 - y^2 + 8x - 2y + 6 = 0$ B. $4x^2 - 3y^2 - 32x + 6y + 73 = 0$

TION III ECCENTRICITY

erbola--definition 3: A hyperbola is the locus of points in a plane such that the ratio of the distance to a fixed point to the distance to a fixed line is some constant e, e > 1. This ratio e is called the eccentricity of the hyperbola.

ly the diagram below:



efinition:

$$\frac{VF}{VE} = \frac{V'F}{V'T}$$

$$\frac{c-a}{a-g} = \frac{c+a}{a+g}$$

$$(c-a)(a+g) = (c+a)(a-g)$$

$$ac - a^2 + cg - ag = ac - cg + a^2 - ag$$

$$2cg = 2a^2$$

$$g = \frac{a^2}{c}$$

. the ellipse, the equation of the directrix is $x = \pm\frac{a^2}{c}$.

Due to symmetry, the hyperbola has two directrices. They are $x = \pm \frac{a^2}{c}$.

When the hyperbola has a vertical axis or if the center is not at (0,0), the necessary adjustment must be made.

For $\frac{y^2}{b^2} - \frac{x^2}{a^2} = 1$, the equation of the directrices is $y = \pm \frac{b^2}{c}$.

To find the value of the eccentricity:

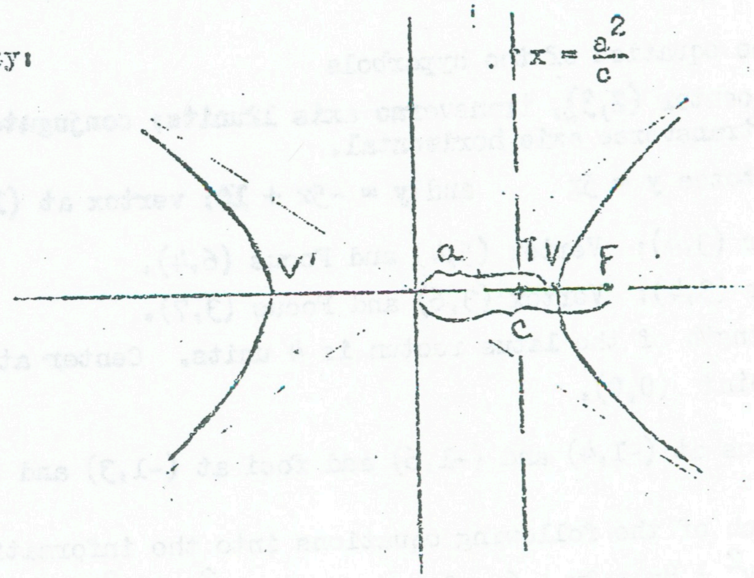
$$\frac{VF}{VT} = e$$

$$\frac{c - a}{a - \frac{a^2}{c}} = e$$

$$\frac{c(c - a)}{ac - a^2} = e$$

$$\frac{c(c - a)}{a(c - a)} = e$$

$$\frac{c}{a} = e$$



EXERCISE 3

1. Write the equation of the hyperbola with focus at (5,0), directrix at $x = \frac{9}{5}$, and eccentricity $\frac{5}{3}$.

2. Write the equation of the hyperbola with focus $(0,\sqrt{41})$, eccentricity $\frac{\sqrt{41}}{5}$, and directrix $y = 25/\sqrt{41}$.

3. Find the equation of the hyperbola whose center is at (-2,-2), the equation of the directrix is y = 1, and the eccentricity is 1.6. What are the equations of the asymptotes?

4. Write the equation of the hyperbola with foci at (-5,0), (5,0), and eccentricity 5/3.

5. Find the foci, equations of the directrices, eccentricity, and equation of of the transverse axis of the hyperbola with equation: $\frac{(x - 3)^2}{12} - \frac{(y + 5)^2}{24} = 1$.

6. Find the equation of the hyperbola whose center is at (3,-1), the equation of a directrix is x = 5, and the eccentricity is 3/2.

The foci of a hyperbola are at (0,3) and (0,-3) and the hperbola passes through the point (0,2). Find the eccentricity and the equations of the directrices of the hyperbola.

ALYTIC GEOMETRY STUDENTS: - Review the L.A.P. Take the trial Run and then take the test on this L.A.P.

TH ANALYSIS STUDENTS GO ON TO SECTION IV

CTION IV. FUN STUFF REVISITED

in previous L.A.P.'s, this section does not really involve any new concepts. re we simply ask you to think a bit more deeply about the properties of the perbola. Go to it!

. The equations of the asymptotes of a hyperbola are $x - 2y = 0$ and $x + 2y = 0$. The hyperbola passes through the point (2,0). What is the equation of the hyperbola?

. Find the coordinates of the center, the axes, and the eccentricity of the hyperbola $9x^2 - 4y^2 - 90x - 24y + 153 = 0$.

. Write the equations of the conjugate hyperbolas which share the asymptotes $4x + 3y - 11 = 0$ and $4x - 3y - 5 = 0$. Let the transverse axis of one of the hyperbolas be of length 6.

. Write the equation of the hyperbola with foci (-2,4) and (6,4) and the slope of one asymptote $\frac{3}{4}$.

5. Write the equation of each of the following hyperbolas. Note: the directrix is not perpendicular to an axis so the equation will not be of the standard form.

A. Directrix $3x + 4y - 2 = 0$, focus (5,5) and eccentricity 2.

B. Directrix $5x - 12y = 0$, focus (0,10), and eccentricicy 1.4.

C. Directrix $4x - 3y + 5 = 0$, focus (0,0), and eccentricity 4.

6. Find the points of intersection of the hyperbolas:

$$x^2 - 2y^2 + x + 8y - 8 = 0 \quad \text{and} \quad 3x^2 - 4y^2 + 3x + 16y - 18 = 0$$

7. If the transvers axis and conjugate axes are congruent, the hyperbola is said to be equilateral. Show that the eccentricity of an equilateral hyperbola is $\sqrt{2}$.

8. Show that the product of the distances of any point on a hyperbola from the asymptotes is a constant.

THE END   REVIEW THE L.A.P.   TAKE THE TRIAL RUN.   TAKE THE TEST.

Hyperbola L.A.P. Answers

EXERCISE 1

1. A. (a) 8; (b) 6; (c) $(\pm 5,0)$.   B. (a) 10; (b) 24; (c) $(\pm 13,0)$.
   C. (a) 8; (b) 6; (c) $(0,\pm 5)$.   D. (a) 14; (b) 4; (c) $(0, \pm\sqrt{53})$
   E. (a) 10; (b) 2; (c) $(\pm\sqrt{26},0)$.   F. (a) 16; (b) $2\sqrt{39}$; (c) $(\pm\sqrt{103},0)$
2. A. $\frac{x^2}{25} - \frac{y^2}{11} = 1$ B. $\frac{x^2}{25} - \frac{y^2}{36} = 1$ or $\frac{y^2}{25} - \frac{x^2}{36} = 1$
   C. $\frac{5x^2}{64} - \frac{5y^2}{16} = 1$ D. $\frac{x^2}{16} - \frac{y^2}{48} = 1$;   E. $\frac{y^2}{100} - \frac{x^2}{16} = 1$
   F. $\frac{y^2}{84} - \frac{x^2}{16} = 1$
3. A. $y^2 - x^2 = 1$ B. $y^2 - \frac{x^2}{2} = 1$ C. $x^2 - \frac{y^2}{6} = 1$

EXERCISE 2

1. A. (a) $(6,3)$ and $(-2,3)$; (b) $(2 \pm 2\sqrt{5}, 3)$; (c) $\frac{y-3}{x-2} = \pm\frac{1}{2}$; (d) 2
   B. (a) $(-5,2)$ and $(-5,-4)$; (b) $(-5,-1 \pm 3\sqrt{2})$; (c) $\frac{y+1}{x+5} = \pm 1$; (d) 6
   C. (a) $(2,-6)$ and $(6,-6)$; (b) $(4 \pm \sqrt{5}, -6)$; (c) $\frac{y+6}{x-4} = \pm\frac{1}{2}$; (d) 1
   D. (a) $(3 \pm 5\sqrt{5}, -3)$; (b) $(3 \pm 5\sqrt{6}, -3)$; (c) $\frac{y+3}{x-3} = \pm\frac{1}{\sqrt{5}}$; (d) $2\sqrt{5}$.

XERCISE 2 answers continued

. A. $\frac{(x-2)^2}{36} - \frac{(y-3)^2}{4} = 1$ B. $\frac{(y-5)^2}{25} - (x-1)^2 = 1$

C. $\frac{(x-3)^2}{4} - \frac{(y-4)^2}{5} = 1$ D. $\frac{(y-4)^2}{4} - \frac{(x-3)^2}{5} = 1$

E. $\frac{4(x-3)^2}{9} - \frac{(y-3)^2}{3} = 1$ OR

E. $\frac{4(y-3)^2}{9} - \frac{(x-3)^2}{3} = 1$ F. $(y-5)^2 - \frac{(x+1)^2}{3} = 1$

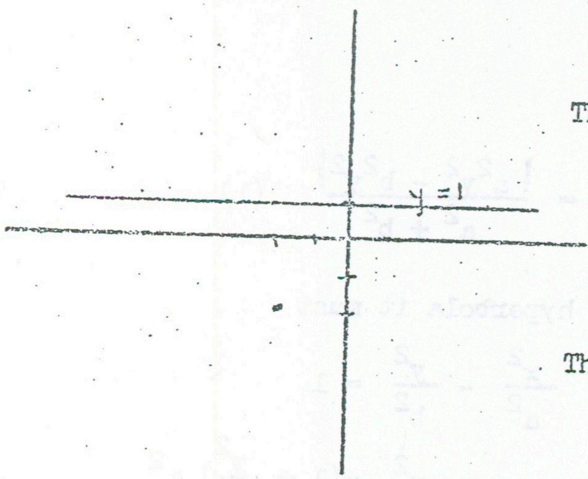
, A. $\frac{(y+1)^2}{3} - \frac{4(x+1)^2}{3} = 1$ B. $\frac{(y-1)^2}{4} - \frac{(x-4)^2}{3} = 1$

ERCISE 3

$\frac{x^2}{9} - \frac{y^2}{16} = 1$ 2. $\frac{y^2}{25} - \frac{x^2}{16} = 1$

This is a toughy so we shall give more than just the answer!

The equation will be of the form: $\frac{(y+2)^2}{b^2} - \frac{(x+2)^2}{a^2} = 1$



The equation of a directrix: $y = -2 + \frac{b^2}{c}$

$1 = -2 + b^2/c$

$3 = b^2/c$

$c = b^2/3$

The eccentricity: $1.6 = \frac{c}{b}$

$c = \frac{8b}{5}$

Hence: $b = \frac{24}{5}$; $c = \frac{192}{25}$

hen using the fact that $a^2 + b^2 = c^2$ we find that $a^2 = \frac{22464}{625}$

he solution is: $\frac{25(y+2)^2}{576} - \frac{625(x+2)^2}{22464} = 1$; Asymptotes $\frac{y+2}{x+2} = \pm \frac{5}{\sqrt{39}}$

$\frac{x^2}{9} - \frac{y^2}{16} = 1$

5. Foci: $(9,-5)$ and $(-3,-5)$.

Directrices: $x = 5$ and $x = 1$; Eccentricity: $\sqrt{3}$; Axis: $y = -5$.

$\frac{(x-3)^2}{9} - \frac{4(y+1)^2}{45} = 1$ 7. Eccentricity: $\frac{3}{2}$; Directrices: $y = \pm \frac{4}{3}$.

EXERCISE 4

1. $\frac{x^2}{4} - y^2 = 1$

2. Center $(5,-3)$; Axis: $y = -3$; $e = \frac{\sqrt{13}}{2}$

3. $\frac{(x-2)^2}{9} - \frac{(y-1)^2}{16} = 1$ and $\frac{(y-1)^2}{16} - \frac{(x-2)^2}{9} = 1$

4. $\frac{25(x-2)^2}{256} - \frac{25(y-4)^2}{144} = 1$

5. A. $25(x-5)^2 + 25(y-5)^2 = 4(3x+4y-2)^2$

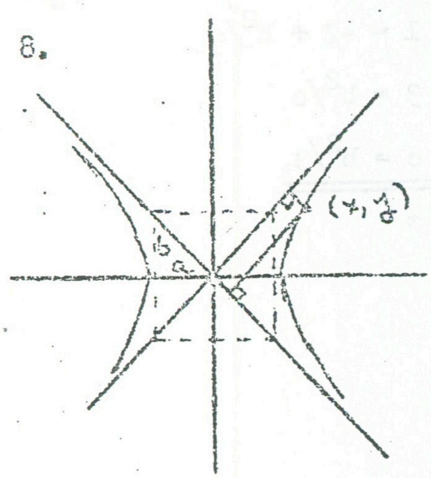
   B. $65^2 x^2 + 65^2(y-10)^2 = 49(5x-12y)^2$

   C. $25x^2 + 25y^2 = 16(4x-3y+5)^2$

6. $(1,3)$; $(-2,3)$, $(1,1)$, and $(-2,1)$

7. --

8.



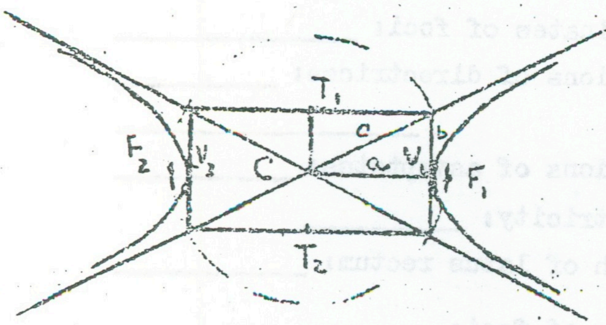
$$\frac{|-bx+ay|}{\sqrt{a^2+b^2}} \cdot \frac{|bx+ay|}{\sqrt{a^2+b^2}} = \frac{|a^2y^2 - b^2x^2|}{a^2+b^2} \qquad (1)$$

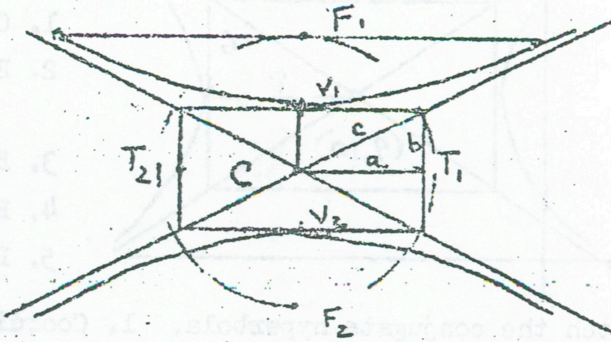
since (x,y) is on the hyperbola it must satisfy the equation: $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

$$x^2 = \left(1 + \frac{y^2}{b^2}\right) a^2$$

Substitute for x^2 in (1) and show that the constant is : $\frac{b^2a^2}{a^2+b^2}$

THE ANATOMY OF A HYPERBOLA





For C at (0,0), the equation of the above is: $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

For C at (0,0), the equation of the above is: $\frac{y^2}{b^2} - \frac{x^2}{a^2} = 1$

C: Center. $CF_1 = c =$ focal distance.

$CV_1 =$ Semi-transverse axis. $CT_1 =$ Semi-conjugate axis.

The intersecting lines are asymptotes. The asymptotes have a slope of $\pm \frac{b}{a}$.

Latus rectum: A chord of the hyperbola which contains the focus and is perpendicular to the transverse axis.

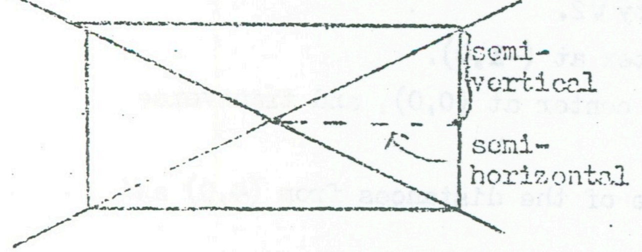
Eccentricity: $\frac{c}{\text{semi-transverse axis}}$

Length of latus rectum: $\frac{2(\text{semi-conjugate axis})^2}{\text{semi-transverse axis}}$

Directrices: Horizontal hyperbola, center at (0,0): $x = \pm \frac{(\text{semi-trans. axis})^2}{c}$

Vertical hyperbola: center at (0,0): $y = \pm \frac{(\text{semi-trans. axis})^2}{c}$

Asymptotes: Every hyperbola has a defining rectangle.



Equation of asymptotes:

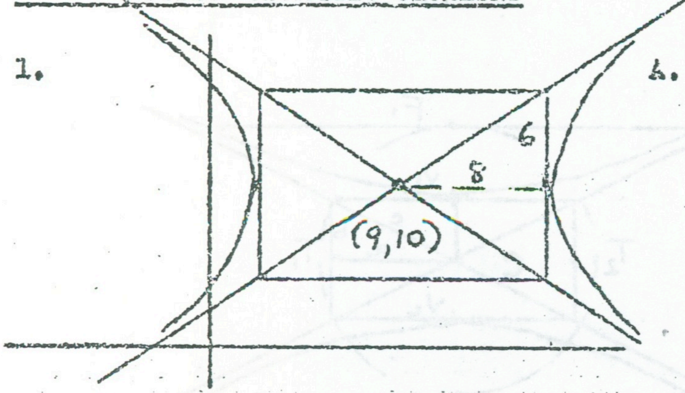
Center: (0,0): $\frac{y}{x} = \frac{\text{semi-vertical}}{\text{semi-horizontal}}$

Center: (h,k): $\frac{y-k}{x-h} = \frac{\text{semi-vert.}}{\text{semi-horiz.}}$

Conjugate hyperbolas share the same center and the same asymptotes.

MATH ANALYSIS--ANALYTIC GEOMETRY HYPERBOLA TRIAL RUN

1.



A. Equation of the hyperbola: ________

1. Coordinates of foci: ________
2. Equations of directrices: ________ ________
3. Equations of asymptotes: ________
4. Eccentricity: ________
5. Length of latus rectum: ________

B. Sketch the conjugate hyperbola. 1. Coordinates of foci: ________

2. Equations of directrices: ________
3. Eccentricity: ________ 4. Length of latus rectum: ________

2. Sketch the hyperbola defined by the equation $(y - 4)^2 - 4(x + 1)^2 = 36$.

A. The length of the transverse axis is ________. B. The length of the conjugate axis is ________. C. The eccentricity is ________.

D. The equations of the directrices: ________

E. The equations of the asymptotes: ________

F. The equation of the conjugate hyperbola: ________

3. Find the coordinates of the centers of each of the following hyperbolas:

A. $x^2 - 4y^2 + 2x + 8y - 7 = 0$ B. $5y^2 - 4x^2 + 20y + 8x = 4$

C. $9x^2 - 4y^2 - 18x - 16y + 29 = 0$ D. $x^2 - y^2 = 4$

4. Write the equation of each of the following hyperbolas:

A. Foci at (0,0) and (0,4) and contains the point (12,9).

B. Focus at (1,-3), directrix $y = 2$, and eccentricity 3/2.

C. Center at (2,-3), conjugate axis of length 6 and transverse axis of length $2\sqrt{2}$.

D. Vertices at (2,0) and (-2,0) and eccentricity $\sqrt{2}$.

E. Asymptotes $y = 3x + 1$ and $y = -3x - 5$. Vertex at (-1,4).

F. Length of latus rectum: 6; eccentricity: 2; center at (0,0), and transverse axis on the Y-axis.

G. The locus of points such that the difference of the distances from (4,0) and (-4,0) is always 2.

H. Axes on the coordinate axes and contains the points (-3,4) and (5,6).